UNIFIED MATHEMATICS

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PREFACE

THIS text presents a course in elementary mathematics adapted to the needs of students in the freshman year of an ordinary college or technical school course, and of students in the first year of a junior college. The material of the text includes the essential and vital features of the work commonly covered in the past in separate courses in college algebra, trigonometry, and analytical geometry.

The fundamental idea of the development is to emphasize the fact that mathematics cannot be artificially divided into compartments with separate labels, as we have been in the habit of doing, and to show the essential unity and harmony and interplay between the two great fields into which mathematics may properly be divided; *viz.*, analysis and geometry.

A further fundamental feature of this work is the insistence upon illustrations drawn from fields with which the ordinary student has real experience. The authors believe that an illustration taken from life adds to the cultural value of the course in mathematics in which this illustration is discussed. Mathematics is essentially a mental discipline, but it is also a powerful tool of science, playing a wonderful part in the development of civilization. Both of these facts are continually emphasized in this text and from different points of approach.

The student who has in any sense mastered the material which is presented will at the same time, and without great effort, have acquired a real appreciation of the mathematical problems of physics, of engineering, of the science of statistics, and of science in general.

A distinctly new feature of the work is the introduction of series of "timing exercises" in types of problems in which the student may be expected to develop an almost mechanical ability. The time which is given in the problems is wholly tentative; it is hoped, in the interest of definite and scientific knowledge concerning what may be expected of a freshman, that institutions using this text will keep a somewhat detailed record of the time actually made by groups of their students. The authors invite the coöperation of teachers of ele-

mentary college mathematics in the attempt to secure this valuable information. The authors will make every effort to put information thus secured at the service of the public interested.

In general, the diagrams are carefully drawn on paper with subdivisions of twentieths of an inch. It is expected that this kind of paper will be used as far as possible in the graphical work, as students will be found to acquire rapidly the ability to use intelligently this type of coördinate paper. Considerable attention should be paid by the teacher to the intelligent reading and interpretation of the diagrams which appear in the text, as the student will in this way gain power to handle his own diagrams, and appreciation of the vital importance of the method. The photographic illustrations should also be used in a somewhat similar manner.

The material can be covered without systematic omissions in a course which devotes five hours per week for one year to the study of mathematics. In a four-hour course there are certain omissions which can be made by the teacher at his own discretion; the three chapters on solid analytical geometry are not commonly presented in the ordinary four-hour course; the chapter on "Poles and Polars" may also be omitted. The exercises are so numerous that any teacher can make a selection, which can be varied, if desired, in succeeding years.

No attempt has been made to introduce the terminology of the calculus as it is found that there is ample material in the more elementary field which should be covered before the student embarks upon what may properly be called higher mathematics. However, the fundamental idea of the derivative is presented and utilized without the new terminology.

The authors are greatly indebted to a large number of their colleagues who have been most generous in furnishing real illustrations in various fields. Professor N. H. Williams of the Department of Physics at the University of Michigan has given very pertinent and valuable comment on numerous sections, in addition to furnishing the beautiful oscillograms of alternating currents. Professor W. J. Hussey of the Detroit Observatory furnished the temperature and barometer chart, and has given generously of his time in the discussion of astronomical problems adapted to an elementary text. Professors J. J. Cox, H. E. Riggs, A. F. Greiner, H. H. Higbie, J. C. Parker, Leon J. Makielski, E. M. Bragg, H. W. King, and L. M. Gram of the Department of Engineering, University of Michigan, have given valu-

able advice and suggestions. The diagram illustrating the use of the ellipse in determining the proper amounts of sand and gravel to use from given pits to obtain the best results was furnished by Professor Cox. To Professor Greiner we are indebted for the cut of the six cylinders of an automobile engine, and for criticising the piston-rod motion. To Mr. Makielski, the well-known artist, we are indebted for the drawing of a box which is reproduced. Professor James W. Glover of the Actuarial and Statistical Department, University of Michigan, has read and corrected the material relating to his field. To Professor C. L. Meader of the Department of Linguistics, and to Professors Pillsbury and Shepard of the Department of Psychology, University of Michigan, we are indebted for the tuning-fork records and for the vowel and consonant records. To Professor F. G. Novy of the Hygienic Laboratory we are indebted for certain information concerning bacterial growth. Captain Peter Field, Coast Artillery, U.S.A., has indicated to us certain simple problems connected with artillery work. To Mr. H. J. Karpinski we owe the photographs of the Rialto and the Colosseum. To the Albert Kahn Company of Detroit we are indebted for information concerning details of the Hill Auditorium, and to the Tyrrell Engineering Company of Detroit for permission to reproduce a number of photographs of bridges. We render to these gentlemen and to our colleagues who have been generous in giving time and thought to our inquiries our sincere appreciation for their friendly coöperation. In every field which we touch, we assume full responsibility for all errors, and we shall be grateful to teachers who will assist in removing the inevitable blemishes in a book of this size and character.

The proof has been carefully read by Professor E. V. Huntington of Harvard University and by Professor C. N. Moore of the University of Cincinnati. Many blemishes have been removed and many important additions and changes have been made on their advice. Professor J. W. Bradshaw of the University of Michigan and Professor J. D. Bond of the Texas Agricultural College have read the galley proof and given numerous and excellent suggestions. Professor W. W. Beman of the University of Michigan has read the page proof and has made numerous vital corrections and suggestions. Professor J. L. Markley has given advice on the early chapters. To all of these gentlemen we acknowledge our real indebtedness.

In putting the work through the press, the responsible editorship has been placed in the hands of Professor Karpinski, as the exigencies

of time and space — Texas to Michigan to New York to Boston would have delayed the book for a full year with a divided responsibility. Certain chapters, including the chapter on the applications of the conic sections, the chapters on the sine curve, on the growth curve and on complex numbers, the treatment of solid analytics, the tables and most of the problems, are due entirely to Professor Karpinski.

The drawings have been made at the University of Michigan, chiefly by Mr. E. T. Cranch, an engineer now in the service, U.S.A. Most of the photographs are by Miss F. J. Dunbar of the University of Michigan Lantern Slide Shop, and a few are by Mr. G. R. Swain.

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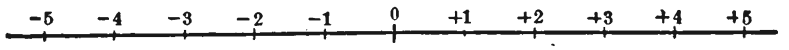
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UNIFIED MATHEMATICS

CHAPTER I

NUMBERS OF ALGEBRA



1. Representation of points on a line. — With any given unit of length and a fixed point of reference, called the origin, the points upon a given line are located by numbers. The unit of length and parts thereof are laid off in both directions from the origin to locate further points. To each point corresponds one number (a symbol) and only one, and to each number corresponds one and only one point. We call this a one-to-one correspondence. To any point upon the line of reference corresponds evidently another point symmetrically placed with respect to the origin. This symmetry is indicated in the symbols by using the same set of symbols twice, distinguishing by two "quality" signs + and −. All points on one side of 0 have the + sign prefixed to the symbols designating them, while the corresponding points on the other side take the same symbols, prefixing the negative sign. Thus $+a$ and $-a$ represent symmetrically placed points on the scalar line. The line of reference is now called a *directed* line. Of two numbers represented by points on this line, the one represented by that one of the two points which lies to the right hand is called the greater. Such a line is the line on an ordinary thermometer; to each number, then, there corresponds further a certain temperature. Thousands of physical and material interpretations of

the points upon such a line and the corresponding numbers are possible.

This scalar line, as the above is termed, is not necessarily a straight line. Thus the equator is a scalar line as it is represented upon any globe, with the zero at the intersection with the meridian of Greenwich, and distances given in degrees, each representing $\frac{1}{360}$ part of the equatorial circumference of the earth; + and − are represented on this line by E. and W.

PROBLEMS

1. Interpret a scalar line as representing distance upon the main line of the Michigan Central Railroad from Detroit, east and west.

2. What is the significance of points to the left of the origin when the line represents your bank account?

3. Interpret the line as representing weights.

4. Interpret the line as the prime meridian. What length is represented by 1° (circumference of earth is 25,000 miles)?

5. Interpret the scale as representing percentage of fat in foods.

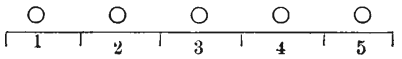
6. Represent the Fahrenheit scale on one side of such a line and the Centigrade upon the other, making the zero and the 100° of the Centigrade scale fall upon the 32° and 212° of the Fahrenheit; note that for a convenient total length 20° Fahrenheit may be taken as corresponding to 1 centimeter or to one half an inch.

2. Real numbers; positive integers. — The symbols representing the points upon a line, as above, are called real numbers. Elementary algebra is largely a study of such numbers, combined according to certain rules. The rules of the game of algebra, as we may term it, can be studied entirely apart from any physical application, but the study is of fundamental importance because of the part which algebraic numbers play in the sciences. However, a knowledge of the laws

of algebra, apart from the applications, is necessary to enable one to apply the numbers effectively to physical problems.

The real numbers are sub-divided into *positive* and *negative* numbers; another classification is into *rational* and *irrational* numbers, the rational numbers being further sub-divided into *integers* and *fractions*. Numbers are represented by the letters $a, b, c, \cdots x, y, z$, etc.

Integers were undoubtedly conceived long before man began to write. The idea of an integer involves the notion of a group of individual objects, and of one-to-one correspondence. The idea or notion which is common to all groups of objects which can be placed in one-to-one correspondence with the objects of a given group is called the number of the given group of objects. Thus the pennies ○○○○○ can be placed in one-to-one correspondence with some segments of our line, or with the group of symbols which correspond to these



segments, or with the individuals of any one of infinitely many other groups, of number *five*, which have the one common property that they can be placed in one-to-one correspondence with each other. The definition is recent; the idea is old. One-to-one correspondence appears frequently in physical problems, as in the one-to-one correspondence between degrees Centigrade and degrees Fahrenheit above.

Integers can be used to represent segments of our line of reference, from O as reference point, with some length as unit of measure (or as individual of the group). The extremity farthest from O is marked with the integer corresponding to the number of the segments between that point and O. Evidently certain groups of segments include as subgroups other groups of segments. The number of the including group is called greater than the number of any included group; the included group is smaller, and its number is less than the number of the including group. Thus the group

called eight, 8, has the smaller sub-groups, 1, 2, 3, 4, 5, 6, and 7.



3. Positive integers; fundamental laws, definitions, assumptions, and theorems. — Given two positive integers, a and b, the single group composed of the individuals from two distinct groups of objects, represented by a and b respectively, is represented by another number x, the *sum* of a and b, which latter we term *summands.* The process of finding such a number is called *addition*, and is indicated by writing the sign + between the two given numbers a and b. By the sum of three numbers is meant the number obtained by adding the third to the sum of the first two, and similarly for more numbers than three. The following are assumptions and theorems concerning integers.

I. $x = a + b$, given two integers, the sum exists.

II. $a + b = b + a$, addition is commutative, *i.e.* the order of addition is immaterial.

III. $a + b + c = (a + b) + c = a + (b + c)$; the associative law for addition.

IV. $x + b = a$. Given the sum a and one of the summands, the other summand exists: x is the number which added to b gives a. This defines the operation of subtraction, which is represented by the sign −, to be placed between the sum and the given summand, as in $a - b = x$.

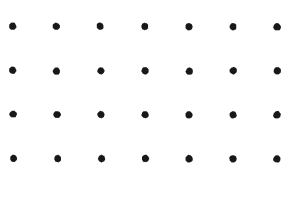
By definition, $(a - b) + b = a$, and for the present this has meaning only when b is less than a; a is termed *minuend*, b is termed *subtrahend*, and $a - b$ is the *remainder.*

Thus, given $x + 2 = 7$, x is evidently 5, as one remembers that in the operation of addition 5 added to 2 gives 7. Given $x + 2 = 2$, or $x + 2 = 1$, we have, at this stage of development, no number x which satisfies the given condition.

V. If $a + c = b + c$, $a = b$, and conversely.

The converse is equivalent to the axiom, if equals be added to equals the sums are equal.

VI. $x = a \cdot b$. Suppose that each individual of a group of a objects consists of b individuals of another type, *e.g.* 4 rows, each of 7 dots, then the single group consisting of all the second type of individuals involved is called the product of a and b. The operation is called *multiplication* and is represented by the sign $\times$ between a and b, or by a period (slightly elevated) between a and b, or by simple juxtaposition of the two numbers, a and b, called factors; a is termed *multiplier*, and b is *multiplicand.*



VII. $a \cdot b = b \cdot a$, the commutative law for multiplication, evident from the figure.

VIII. $a \cdot b \cdot c = (a \cdot b) \cdot c = a\,(b \cdot c)$, the associative law for multiplication.

IX. $a(b + c) = ab + ac$, multiplication is distributive with respect to addition. This corresponds precisely to our ordinary method of multiplication.

X. $b \cdot x = a$. Given the product a, and one factor b, x is defined by this relation as the number which multiplied by b gives a. This operation is limited *when dealing with integers* to numbers a and b, which are so related that a is one of the products obtained by multiplying b by an integer.

The process of finding x is called *division*, and is represented by the sign $\div$, or by placing a over b, $b \cdot \frac{a}{b} = a$; b is termed the *divisor*, a is the *dividend*, and x is termed the *quotient.*

These laws concerning positive integers constitute simply a restatement of facts with which the student is familiar. The four fundamental operations to this point have been confined to the field of positive integers; evidently the operation of division when b is the divisor applies only to those positive integers which are multiples of b. Similarly the operation of subtraction of b from a is limited to integers so related that

$a > b$. We extend our field of numbers by removing these limitations. Thus if you wish to have a number x which multiplied by 8 gives 5 you do not find it among the positive integers; you may then decide to create such a number, calling it $\frac{5}{8}$, the two symbols indicating the definition and genesis of the new number. Such extensions of the number field are briefly indicated in the next section.

4. Rational numbers; zero, fractions, and negative integers. — These fundamental equalities and definitions from I to X are now extended by removing all limitations (except one, as noted below) upon the numbers, a, b, c, and x. Note that only the operations of addition, subtraction, multiplication, and division are included at this point.

Extension of IV. $x + b = a$, when $b \neq a$ defines zero, written 0. By definition then, $0 + a = a$. $x + b = 0$, defines a negative number which is written $-b$. The negative here is a sign of quality; by definition $-b$ is the result of subtracting b from 0, and $-b + b = 0$.

$x + b = a$, when $b > a$, defines the negative number $a - b$, which is the negative of $b - a$.

Subtracting a negative number can now be shown to be equivalent to adding a positive number, and similarly the other rules of elementary algebra relating to the addition and subtraction of positive and negative quantities. That the product of two numbers with like signs is positive and the product of two numbers with unlike signs is negative follows from the above development.

A negative number $-a$ is placed in our line of reference symmetrically to the corresponding positive number a, with respect to the origin; of two negatives, the one toward the right is called the greater.

Extension of X. $b \cdot \frac{a}{b} = a$, for all values of b except 0.

This extension of $\frac{a}{b}$ to mean a number which multiplied by

b gives a introduces new numbers of the type $\frac{a}{b}$, rational fractions in which a and b are positive or negative integers. Integers are included in this definition if b is a factor of a or if b is equal to one. Division by 0 is explicitly excluded.

A rational number is any number which can be expressed as the quotient of two integers, the denominator not to be zero.

All of our rules for operating with fractions follow from the definition of $\frac{a}{b}$ and from the preceding development. Thus $\frac{-a}{b} = -\frac{a}{b} = \frac{-a}{b}$. Further, by definition

$\frac{a}{b} > \frac{c}{d}$, when both are positive if $ad > bc$;

$\frac{a}{b} = \frac{c}{d}$, when both are positive if $ad = bc$; and

$\frac{a}{b} < \frac{c}{d}$, when both are positive if $ad < bc$.

Positive fractions can thus be arranged in a determined order upon our line of reference; the value of the fraction determines the position and a graphical method of locating $\frac{a}{b}$ on the scalar line is indicated in the next section; negative fractions are placed symmetrically to the corresponding positive fractions, with respect to the origin.

Of any two rational numbers a and b, a is greater than b ($a > b$) if $a - b$ is positive; for when $a - b$ is positive, a positive length must be added to b to give a, and consequently a must lie to the right of b. If on the line of reference two points x_1 and x_2 are taken (fixed points), $x_2 - x_1$ gives the distance from the first point to the second; this expression is positive if $x_2 > x_1$, and negative if $x_2 < x_1$.

That there is a distinction between + and − used as signs of operation, as with positive integers in the preceding section,

and + and − used as quality signs is apparent. Thus $-b$ may indicate that b is to be subtracted from some preceding number, or $-b$ may indicate that the distance b is taken on the negative side of the origin. The fact that $a+(-b)$, the addition to a of negative b, gives the same result as subtracting b from a, or $a-b$, is readily shown by the graphical method of section 9 below. This type of relationship obviates any need for careful distinction between the two possible meanings of these signs and makes separate symbols not necessary. When no sign is used with a number symbol the + sign is understood.

EXERCISES

1. Explain the distinction between $\frac{-3}{7}$ and $-\frac{3}{7}$; between $\frac{3}{-7}$ and $\frac{-3}{7}$.

2. Write $\frac{5}{3-x}$ in the three forms corresponding to $\frac{-a}{b}$, $-\frac{a}{b}$, and $\frac{a}{b}$.

3. Is $-3>-2$? Which is greater, 0 or -3? Explain.

4. What is the difference between 4 and -3? 4 and 3? 4 and 11?

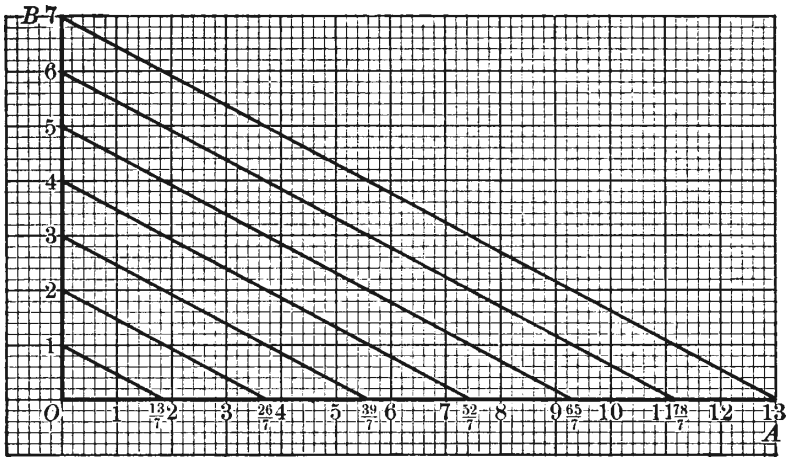
5. What fundamental law is assumed in the common process of multiplication, *e. g.* as in 325 by 239 and also $x-7$ by $x-2$? Is there a corresponding assumption in division?

6. Which is greater, $\frac{1}{2}$ or $-\frac{2}{3}$? Is $\frac{17}{19}$ greater or less than $\frac{33}{38}$? Explain.

7. What is the product of 0 by 7; by -8; by 3; by $\frac{5}{17}$? If a product is zero, what limitation is imposed upon the factors?

5. Representation of a rational number, $\frac{a}{b}$. On cross-section paper any rational fraction can be represented, using ruler and compass. Using 5 divisions to represent unity, each division

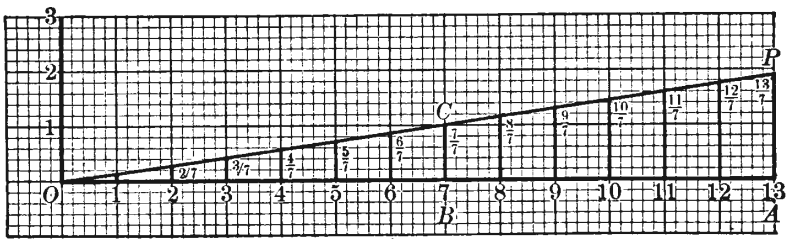
represents $\frac{1}{5}$ of a unit. To represent $\frac{13}{7}$, one measures off 13 units, OA, on the line of reference, and 7 units, OB, on a second line through the origin (for convenience, use cross-sec-



Graphical division

$\frac{1}{7}$ to $\frac{6}{7}$ of 13 represented on horizontal line of reference.

tion paper). Connect the ends, AB, and through the point U, one unit from O on the second line, draw a line parallel to AB. The intersection point on the reference line represents the fraction $\frac{13}{7}$. Similarly any fraction $\frac{p}{q}$ can be represented. The series of parallels to AB through the first 7 unit points on the vertical axis will cut off (plane geometry theorem) 7 equal parts of 13 on the horizontal axis.

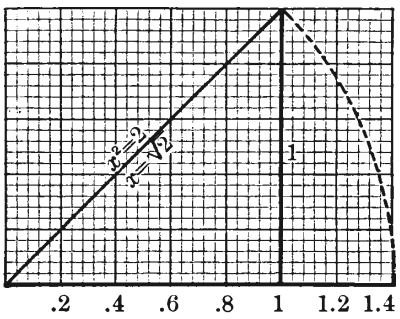


Graphical division

$\frac{1}{7}$ to $\frac{13}{7}$ expressed decimally.

On cross-section paper a somewhat better method of indicating any quotient $\frac{a}{b}$ is to move out on the line of reference b units and up 1 unit; connecting this point with the origin O gives a straight line which can be used to read the desired quotients. Thus, since $OB = 7$ units, $BC = 1$, and $OA = 13$, it follows that $\frac{AP}{BC} = \frac{OA}{OB}$, or $\frac{AP}{1} = \frac{13}{7}$, whence AP equals $\frac{13}{7}$. To obtain $\frac{8.5}{7}$, you find the point 8.5 units from O, and the vertical distance to the oblique line represents $\frac{8.5}{7}$, or 1.2.

6. Irrational numbers. $x^2 = 2$ is a simple and familiar illustration of a relation which is not satisfied by any rational number, $\frac{a}{b}$, with a and b integers; geometrically, the diagonal of a square with side unity is not represented by any rational number. If you wish the length of this diagonal for any practical purpose, you use $\frac{14}{10}$, or $\frac{113}{80}$, or $\frac{141}{100}$, or $\frac{17}{12}$, or $\frac{1414}{1000}$, or $\frac{141142}{100000}$. The carpenter uses 1 foot 5 inches, or $1\frac{5}{12}$ feet, in the diagonal for every foot of side, with an error of $\frac{3}{10}$ of one per cent. The series of rational numbers 1, $\frac{14}{10}$, $\frac{141}{100}$, $\frac{1414}{1000}$, $\frac{141142}{100000}$, which can be indefinitely extended always increasing, and the series, always decreasing, 2, $\frac{15}{10}$, $\frac{142}{100}$, $\frac{1415}{1000}$, $\frac{141143}{100000}$, with a constantly decreasing difference of limit 0 between corresponding terms, together define the irrational number called the square root of 2. No rational number satisfies the relation; no number $\frac{a}{b}$ is at the end of either series, but either series determines a definite point on

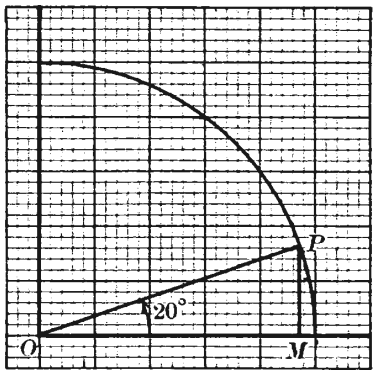


Graphical representation of $\sqrt{2}$

our line, and algebraically defines our number, which we will call the square root of 2.

Proof of the irrationality of $\sqrt{2}$. — Assume that $\sqrt{2} = \frac{p}{q}$, a rational fraction in lowest terms, with p and q integers. Both p and q cannot be even numbers, either p is odd or q is odd. If p is odd, squaring and clearing of fractions, $2q^2 = p^2$; but p is odd and you have an even number equal to an odd number. Hence p cannot be an odd number. Now assume that p is even and that q is odd, and further let $p = 2m$. Then $2q^2 = 4m^2$, $q^2 = 2m^2$, and again we have an odd number equal to an even number. Our assumption that $\sqrt{2} = \frac{p}{q}$ leads to an absurdity, that an odd number equals an even number.

Describe about the origin with a radius 10 a circle, and using a protractor measure an angle of 20 degrees. The length of the perpendicular and the part cut off by the perpendicular from the end of this line are definite and precise points which can be computed to any degree of accuracy desired. No rational number represents these lengths, which are trigonometric irrationalities. A series of constantly increasing rational numbers can be found, such that there is no greatest of the series, to represent lines which are always shorter than the given line; and another series of terms constantly decreasing, but approaching to the terms of the first series, can be found. No largest number can be found in the first series and no smallest in the second; both sequences together define, we may say, an irrational number.



A trigonometric irrational

1. Write 6 terms of the decreasing series defining $\sqrt{2}$; $\sqrt{3}$; $\sqrt{5}$.

2. How is the series for defining the length of the circumference of a circle obtained? What assumption is made?

3. Find the series defining the square root of 3673 by successive approximations.

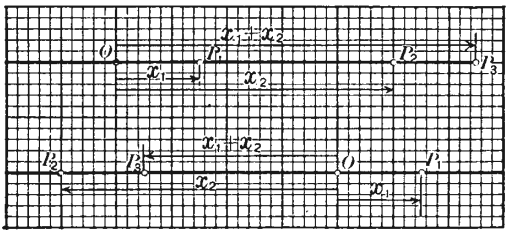
4. Inscribe a square and an equilateral triangle in the circle of radius 10; find the sides.

7. Constants and variables. — Every fixed point on the line of reference is at a fixed distance from the point of reference, O; the distance is constant. We can think of a point as moving on the line OX in either direction. The distance from O then varies and we speak of the distance as a variable. Thus, also, the price of wheat during a term of years or in different parts of the world is a variable; the weight of an animal at different ages is a variable. We think of the variable quantity as taking a series of values under diverse conditions. We can represent the variable distance of a point from O on our line by the single letter x, which may then in the various possible positions on the line be thought of as positive, or negative, or zero, as rational or irrational. This letter x represents then a variable quantity and is essentially a number, subject to all the operations on algebraic quantities as noted above. In general, we designate the variable point by the single letter P, the distance from O is OP, of which the length and direction from O is indicated by the number or variable x. A point on the line $X'X$ is represented by a single letter x, called the *abscissa* of the point. The fixed points on the line are frequently represented by the letters a, b, c, d, $\cdots$ or by x_1, x_2, x_3, $\cdots$, each of which may represent any point upon the line.

8. Historical note. — Modern algebra with the systematic employment of literal coefficients, letters to represent general constants, was introduced by the great French mathematician

and statesman, François Viéte (1540–1603); Viéte used the consonants to represent known quantities and vowels to represent unknowns, using capitals for both. The use of the symbols of operation in equations dates also from about the same time; our equality sign was introduced by Robert Recorde, an English physician and mathematician, whose "Whetstone of Witte," 1557, is the first treatise in the English language on algebra. The + and − symbols are due to a German, Widmann, and date from 1489.

9. Geometrical equivalents of the four fundamental operations. *a.* **Addition.** — The operation of addition of x_1 and x_2 is represented graphically by placing the length OP_2 upon the line



$OP_3 = OP_1 + P_1P_3 = OP_1 + OP_2 = x_1 + x_2.$

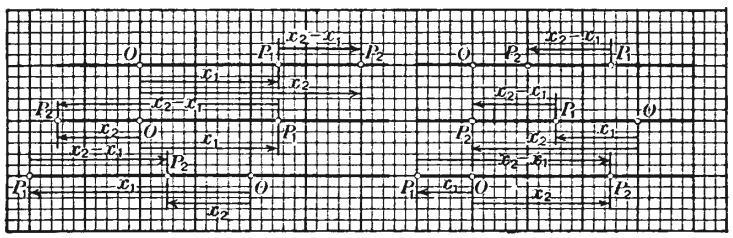
from the point P_1 in the direction of OP_2. Physically, addition is the result in general of two different causes. Thus a weight of 3 pounds + a weight of 5 pounds; a vertical velocity due to the action of gravity (on a falling body) + a vertical velocity due to some other force; a transportation (translation) from one point to another + another translation in the same direction; two successive rotations of a wheel about its axis; these are familiar examples of addition.

b. **Subtraction.** $OP_1 + P_1P_2 = OP_2$; $P_1P_2 = OP_2 - OP_1$.

Whatever the relative positions on the line OX of P_1 and P_2, with respect to the position of O, $OP_1 + P_1P_2 = OP_2$, all of these representing directed line segments. In words, the equality $P_1P_2 = OP_2 - OP_1$ states that the distance from any point P_1 on a directed line to a second point on the line is given by the abscissa of the second point minus the abscissa of

the first, with respect to any third point O, on the line, as origin.

If we represent the distance from P_1 to P_2 by the letter d, we have $x_1 + d = x_2$, or $d = x_2 - x_1$. Subtraction is represented by the distance from the first point to the second point, which



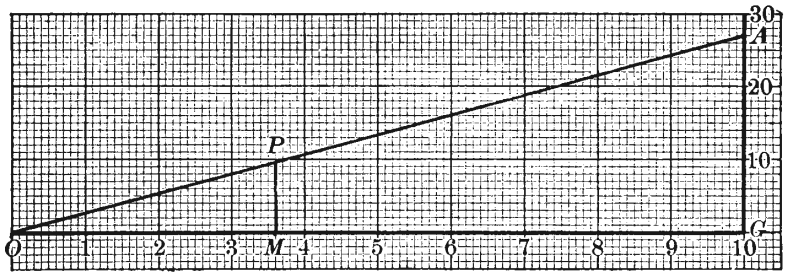
$OP_1 + P_1P_2 = OP_2.$

Fundamental property of any three points on a directed line

equals $OP_2 - OP_1$. Since in physical problems the distance represents the rise or fall in numerical value of physical values, graphical subtraction is more frequently noted than addition. We say the temperature has risen 10 degrees or fallen 10 degrees, having in mind the original and the final reading; the iron bar has expanded an inch, has a precise and valuable physical meaning when the initial and the final length of the bar are known. The operation involved in measuring the expansion is strictly subtraction.

c. **Multiplication and division.** — Graphical multiplication and division upon cross-section paper, involving theorems concerning similar triangles, lends itself to numerical work where great accuracy is not necessary. Some convenient length is taken to represent 1 unit on the horizontal scale and a convenient length as unit on the vertical scale; for three-place accuracy a sheet of cross-section paper considerably larger than the page of this book would be necessary. If multiples and quotients of 27 (2.7 or 270 or 2700 ...) are desired, take the unit on the horizontal scale so as to use 8 to 10 inches to represent the numbers up to 10; on a vertical line above the 10 on the horizontal scale represent 27. On our

diagram below we have used 1 horizontal unit to represent 10 on the vertical scale; greater accuracy would be attained by using a longer length as unit on the vertical scale. The tenths of 27 are represented by the vertical lengths from the points 1, 2, 3, 4, ..., 9 on the horizontal scale to the line joining the intersection of the two scales to the point 27 units above the point 10 on the horizontal scale. To multiply 2.7 by 3.6 you find the vertical length above the point 3.6 on the



Graphical multiplication

Tenths and hundredths of 27 represented by the vertical lines.

horizontal scale and this length is 9.7; for 27×36 you interpret this as 970 and similarly for $.27 \times .36$ you interpret this as .097. Evidently $MP : 3.6 = AG : 10$, or $MP : 3.6 = 27 : 10$, whence $MP = 3.6 \times 2.7$. In division the length MP corresponds to the dividend and 27 to the divisor; the quotient is then given as the horizontal distance corresponding to the given dividend (MP). Thus $20 \div 27$ is obtained graphically by finding the horizontal distance at which the line 20 units above our horizontal scale cuts the given oblique line; this appears to be at 7.4 units and must be interpreted as .74; the position of the decimal point must be fixed by the computer. The explanation would be a trifle easier if the line of length 27 were placed directly above the unit distance, but a slight error in the position of the upper point would produce large errors at the right end of the line; a line to be used as indicated must be kept on the paper and it must be drawn between points which are not close together.

PROBLEMS

1. Represent the square root of 2 geometrically, taking 10 quarter-inches as 1. Represent the square root of 3 on coördinate paper, using the same scale. Represent the square root of 5, 6, 7, and 8.

2. On coördinate paper take a horizontal length of 10 half-inches and at the right extremity draw a vertical line 7 half-inches long. Draw the line connecting the extremity of the vertical line to the left extremity of the horizontal line drawn. Use the divisions of the cross-section paper to read tenths of 7. Find $.43 \times 7$. What change must you make to read 4.3×7? or 43×7?

3. Make the vertical line 7.4 in length, and read tenths of 7.4. Find approximately $.83 \times 7.4$. Interpret as 8.3×7.4 and as 83×74.

4. On the oblique line of problems 2 and 3 find where the vertical distance above the horizontal line is 5. How far out is this on the horizontal line? This, read as tenths per half-inch, represents the quotient of $\frac{5}{7}$ and $\frac{5}{7.4}$ respectively. Find the quotient of 5.3 divided by 7. Find 5.3 divided by 7.4.

5. Draw a semicircle on a diameter of 10 half-inches. Note that the perpendicular at any point on this diameter is, by plane geometry, a mean proportional between the segments of the diameter. Read the mean proportional between 1 and 9, as the vertical line drawn at the point on the diameter 4 units out from the center. Read the mean proportional between 2 and 8, similarly; between 3 and 7; between 4 and 6.

6. Regard the preceding circle as having a radius 10 quarter-inches. Find approximately, from it, $\sqrt{19}$, $\sqrt{36}$, $\sqrt{51}$, $\sqrt{64}$, $\sqrt{75}$, $\sqrt{84}$, $\sqrt{91}$, $\sqrt{96}$, $\sqrt{99}$, and $\sqrt{100}$. These are the vertical lengths at the points dividing the diameter in the ratio 19 to 1; 18 to 2; 17 to 3; ··· 10 to 10.

7. Take a circle of diameter 12 half-inches; from it approximate $\sqrt{11}$, $\sqrt{20}$, $\sqrt{27}$, $\sqrt{32}$, $\sqrt{35}$, and $\sqrt{36}$. Note that $\sqrt{20} = 2\sqrt{5}$, $\sqrt{27} = 3\sqrt{3}$, and $\sqrt{32} = 4\sqrt{2}$; from these approximate $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$. See diagram on page 72.

8. On the preceding circle check the geometrical fact that either side of a right triangle is a mean proportional between the whole hypotenuse and the adjacent segment of the hypotenuse cut off by the perpendicular from the vertex of the right angle.

CHAPTER II

APPLICATION OF ALGEBRA TO ARITHMETIC

1. Arithmetic in science. — In science definite progress is usually intimately associated with the arithmetical investigation of the data of the science. Even in medicine, biology, sociology, and chemistry, arithmetic plays a large rôle; in physics, astronomy, and engineering, computation is absolutely essential. The student of mathematics must take account of the numerical side of mathematical work, constantly applying the theoretical work to definite, numerical problems.

The accuracy of computed results in scientific work depends upon the accuracy of the observation measurements upon which these computations are based. This can be taken as almost axiomatic. Whenever numbers are given representing measurements, the computations involving these numbers should not be carried beyond the point which these measurements justify.

2. Arithmetical application of algebraic formulas. — Algebraic formulas and methods can frequently be applied to arithmetical problems with a great saving of labor; practice with numerical examples is absolutely essential for success.

The student is expected to use constantly the rules given in elementary arithmetic for multiplying and dividing by 5, 25, 12.5, $33\frac{1}{3}$, and by these numbers multiplied by integral powers of 10. Thus, to multiply by 5, multiply by 5 as $\frac{10}{2}$, annexing a zero and dividing by 2; similarly, 25, either as multiplier or divisor, is considered as $\frac{100}{4}$, and so with the other aliquot parts of 100 which have been mentioned. It is better, in general, to use these rules with the few numbers mentioned, and their

decimal multiples, rather than to extend this work to other aliquot parts whose occurrence is less frequent.

ILLUSTRATION. — 25×374 is written $4\overline{)37400}^{9350}$; 2.5×375 is written $4\overline{)3750.0}^{937.5}$; $173 \div 250$ is written $.173 \times 4 = .692$; 75×375 is written:

18750 (50 times 375, dividing 375 by 2 and extending 2 places.)
9375 (25 times 375, or $\frac{1}{2}$ of 50 times.)
28125 (75 times 375.)

The four formulas of elementary algebra which enjoy the widest use are undoubtedly:

$$(x + a)^2 = x^2 + 2ax + a^2.$$
$$(x - a)^2 = x^2 - 2ax + a^2.$$
$$(x + a)(x - a) = x^2 - a^2.$$
$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

$(x + a)(x + b)$ gives a simple rule for the product of two "-teens," *e.g.* 19×17.

Thus, $(10 + a)(10 + b) = 10^2 + (a+b)10 + ab$, or $= 10(10 + a + b) + ab$.

Put into words, this formula states that the product of two numbers between 10 and 20 is equal to the whole of one plus the units of the other; this sum is to be multiplied by 10; to this product is to be added the product of the units.

RULE. — *To find the product of two "-teens," add the whole of one to the units of the other and annex a zero; to this number add the product of the units.*

19
17
260
63
323

If x is taken as 20, 30, 40, 50, ..., the corresponding rule for the product of two two-place numbers having the same tens' digit is to add to the one of two numbers the units of the other; the sum is to be multiplied by the tens' digit, and a

zero annexed to the product; to this number add the product of the units.

Thus, $(37 \times 35) = 30 \times 42 + 35 = 1260 + 35$
$- 1295.$

Such products are most easily found, evidently, if the two units' digits sum to 10.

$87 \times 83 = 8 \times 9 \times 100 + 21$
$- 7221.$
$64 \times 66 = 6 \times 7 \times 100 + 24$
$- 4224.$

In mental work with numbers work from *left to right*, and not from right to left, dealing first with the numbers of greater significance.

$(x + a)^2$ and $(x - a)^2$ are particularly useful in the computation of squares of numbers of three places beginning with 1 or 9.

$(10.7)^2 = 100 + 2 \times 10 \times .7 + .49$
$- 114.49.$
$(11.3)^2 = 121 + 6.6 + .09 = 127.69,$
or $- 100 + 26 + 1.69 = 127.69.$
$(1.57)^2 = 2.25 + .21 + .0049,$

where .21 is obtained as $1.5 \times .14$ by the rule for the product of two "-teens."

$(.97)^2 = (1.00 - .03)^2 = 1 - .06 + .0009$
$= .9409.$
$(8.70)^2 = (10 - 1.3)^2 = 100 - 26 + 1.69 = 75.69.$

Frequently it is more convenient to use these formulas rearranged as follows:

$(x + a)^2 = x(\overline{x + a} + a) + a^2.$
$(x - a)^2 = x(\overline{x - a} - a) + a^2.$

Thus, $(84)^2 = 100(100 - 16 - 16) + 16^2$
$- 100(84 - 16) + 16^2$
$- 6800 + 256 = 7056.$
$(25.7)^2 = 20(25.7 + 5.7) + (5\ 7)^2$
$- 628 + 32.49$
$- 660.49.$
$(25.7)^2 = 25(25.7 + .7) + .49 = 25(26\ 4) + .49$
$- 660.0$ (since $25 = \frac{1}{4}\frac{0}{0}$) $+ .49.$

The square of any number between 25 and 75 is obtained from $(x + a)^2$, as follows:

$$(50 \pm a)^2 = 2500 \pm 100\,a + a^2 = 100 \times (25 \pm a) + a^2.$$

Thus,
$$(37)^2 = 2500 - 1300 + 169$$
$$= 1369.$$

RULE. — *To find the square of any number between 25 and 75; find the difference between the given number and 50; add, if the given number is greater than 50, or subtract, if the given number is less than 50, this difference from 25 and annex to this two zeros. Add to this number the square of the difference.*

Thus,
$$(65)^2 = (25 + 15) \times 100 + 15^2$$
$$- 4225.$$

For numbers between 75 and 150 the squares may be obtained as $100(\overline{100 - a} - a) + a^2$ or $(100)(\overline{100 + a} + a) + a^2$, noting that $100 - a$ or $100 + a$ is your given number whose square is sought.

Thus,
$$112^2 = 12{,}400 + 144.$$
$$13.7^2 = 174.00 + 3.7^2 = 174.00 + 13.20 + .49 = 187.69.$$

Frequently, of course, only three or four significant figures are desired, and the methods mentioned give the significant figures first.

$(x + a)(x - a)$ may also be used for squares, thus:

$$x^2 = (x + a)(x - a) + a^2$$
$$(87)^2 = (87 + 13)(87 - 13) + 13^2.$$
$$(2.33)^2 = (2.33 + .17)(2.33 - .17) + .17^2$$
$$- 5.4 + .0289.$$
$$(41.7)^2 = (41.7 + 8.3)(41.7 - 8.3) + (8.3)^2$$
$$- 50 \times 33.4 + 8.3^2$$
$$- 1670 + 68.89$$
$$- 1738.89.$$
$$(41.72)^2 = 1738.89 + (.04)(41\ 7) + .0004$$
$$- 1738.89 + 1.6684 = 1741 \text{ to units, or } 1740.6 \text{ to tenths.}$$

PROBLEMS

1. Multiply mentally 19×18, 17×15, 18×14.

2. Use the rule given above to give the table of 18's from 18×11 to 18×19.

3. Multiply mentally 12×13, 36×34, 45×45, 82×88, 91×99.

4. Multiply mentally 27×25, 34×32, 54×58, 92×98.

5. What is the product of 44×36 or $(40 + 4) \times (40 - 4)$, 58×62, 44×37 or $(40 + 4) \times (40 - 3)$?

6. What are the first three figures of $(114)^2$, $(107)^2$, $(131)^2$, and $(118)^2$? Note $(114)^2$ is $12{,}800 + 14^2$, and the first three figures 129; in $(116)^2$ to 13,200 you must add $(16)^2$, which increases the first 13,200 to 13,400.

7. From the preceding answers in 6 write the first three figures of $(1.14)^2$, $(.107)^2$, $(1.31)^2$, $(1180)^2$

8. Write the squares of 9.7, 88, 940, 8.7, and 9.2.

9. Approximately how much greater is $(9.71)^2$ than $(9.7)^2$? $(88.2)^2$ than $(88)^2$? $(941)^2$ than $(940)^2$? $(8.75)^2$ than $(8.7)^2$? $(9.26)^2$ than $(9.2)^2$?

Note that $(88.2)^2$ differs from $(88)^2$ first by $.4 \times 88$, or by a little more than 35 units; the .04 is usually negligible.

10. Square 43, 47, 52, 63, and 62 by using the difference between these numbers and 50 according to the rule.

11. Using the preceding answers, square 4.3, .47, .052, 630, and 6.2.

NOTE. — Use common sense rules to determine the position of the decimal point.

12. Using the formulas for $(50 \pm a)^2$, $(x \pm a)^2$, $(100 \pm a)^2$ write the following 25 squares. Time yourself on writing simply the answers; the exercise should be completed in 6 minutes.

$57^2 =$	$17^2 =$	$24^2 =$	$33^2 =$	$62^2 =$
$63^2 =$	$42^2 =$	$39^2 =$	$52^2 =$	$67^2 =$
$87^2 =$	$63^2 =$	$59^2 =$	$71^2 =$	$82^2 =$
$43^2 =$	$10.8^2 =$	$21^2 =$	$66^2 =$	$1.9^2 =$
$98^2 =$	$16^2 =$	$92^2 =$	$55^2 =$	$49^2 =$

13. Using the results of the preceding exercise, compute to four significant figures the following squares, timing yourself.

$57.1^2 =$	$17.3^2 =$	$24.5^2 =$	$33.1^2 =$	$62.4^2 =$
$63.2^2 =$	$42.5^2 =$	$39.7^2 =$	$52.9^2 =$	$67.1^2 =$
$87.4^2 =$	$63.3^2 =$	$59.2^2 =$	$71.8^2 =$	$82.9^2 =$
$43.5^2 =$	$10.82^2 =$	$21.4^2 =$	$66.7^2 =$	$1.92^2 =$
$98.6^2 =$	$16.6^2 =$	$92.6^2 =$	$55.3^2 =$	$49.8^2 =$

14. Employing the formula for $(x+a)(x+b)$ write the following products; the exercise should be completed in 6 minutes.

16×19	22×24	32×38	51×52	66×64
15×14	23×26	43×42	33×31	88×82
13×18	24×29	46×44	27×24	97×93
17×12	28×28	54×59	24×22	57×53
16×18	36×33	82×87	17×13	79×71

3. Extraction of roots. — In extraction of square root, the method of successive approximation should frequently be employed.

Thus, $\sqrt{179.63} > 13$ and < 14.

$\sqrt{169 + 10.63} = 13 + a$, wherein a must be a number such that $2a \times 13$ equals approximately 10.6. A glance shows that $.4 \times 13$ equals 5.2, which doubled gives 10.4. Hence, $(13.4)^2 = 169 + 10.4 + .16$.

$$(13.4)^2 = 179.56, \text{ or } 179.63 - .07.$$

$$(13.4 + a)^2 = 179.56 + 2a \times 13.4 + a^2.$$

a now is less than .01 ; hence, a^2 is less than .0001 ; a is to be a number of hundredths or thousandths, evidently, so that $2a \times 13.4$ is approximately .07 ; a is roughly .003, slightly too large.

$$(13.403)^2 = 179.56 + .0804 + .000009$$
$$= 179.6413.$$

The rule is commonly given to take $x \pm \frac{a}{2x}$ as first approximation of $\sqrt{x^2 \pm a}$, wherein a is small as compared with x. The process illustrated follows this rule, but suggests thinking multiplication instead of division. Thus in the square root of 300, as $\sqrt{17^2 + 11}$, approximately $\frac{11}{34}$ is to be added to 17; however it is easier to think $34 \times a = 11$, whence $a = .3$, or not quite .33; trying .32 (since the a^2 term is to be added) gives 17.32 of which the square is

$$289 + 10.88 \text{ (or } .32 \times 34) + (.32)^2 = 299.9829.$$

Similarly, $\sqrt{3000} = \sqrt{3025 - 25} = \sqrt{55^2 - 25} = 55 - a$, wherein a must be a number such that $2 \times 55 \times a$ will give approximately 25; a is evidently .2 to one decimal place or .23 to two; 54.77 is correct to four significant figures as given.

4. Approximate roots. — Another method of approximating square root is to divide the given number by the first approximation, then to use the arithmetic mean of the two numbers as a second approximation. Thus, $179.63 \div 13 = 13.82$; taking $\frac{13 + 13.82}{2}$ as the approximate root gives 13.41 as a second approximation. $179.63 \div 13.41$ gives 13.3952 and the average 13.4026 is within .0001 of the correct value.

Similarly the cube root may be obtained. Thus in 179.63, 5 is the first approximation. $5^2 = 25$; $179.63 \div 25 = 7.2$ nearly. Taking the average of 5, 5, and 7.2 gives 5.7 as second approximation; $(5.7)^2 = 32.49$; $179.63 \div 32.49 = 5.529$; the average of 5.7, 5.7, and 5.529 gives 5.643, which is correct within .001.

PROBLEMS

1. What is the approximate square root of 1.26? 128?
2. Is the square root of 1.35 nearer to 1.16 or to 1.17?

NOTE. $(1.16)^2 = 1.32 + .0256$ and $(1.17)^2 = 1.34 + .0289$.

3. By successive approximations find $\sqrt{1.26}$ to four decimal places and compare with ordinary process of extraction of root.

HINT. — Use 1.12 as first approximation.

4. Find the cube root of 1.26 by this process, using 1.08 as first approximation.

5. Find the approximation to one decimal place of the square roots of 65, 63, 8.30, 8.76, and 27.32.

HINT. — Regard 8.30 as $(3 - x)^2$, whence x must be roughly .12.

6. Write the square roots of the following numbers, correct to 2 decimal places. Time yourself.

9.9	35	65	140	200
16.8	34	68	150	300
17.2	37.2	78	125	10.4
25.8	39.4	85	108	20.8
28.	48	90	112	30.6

5. Synthetic division and remainder theorem. — The ordinary process of division, and particularly the abbreviated process by detached coefficients, can frequently be applied to computation.

$$
\begin{array}{r}
3x^2 + 4x + 15 \\
x - 2 \overline{)\,3x^3 - 2x^2 + 7x - 5} \\
\underline{3x^3 - 6x^2} \\
\quad + 4x^2 \\
\quad \underline{+ 4x^2 - 8x} \\
\qquad + 15x \\
\qquad \underline{+ 15x - 30} \\
\qquad\qquad + 25
\end{array}
$$

Notice that the 3 of $3x^2$ has been written three times; the 4 of the $4x^2$ has been written three times; the 15 of the $15x$ has been written three times. Each of these, 3, 4, and 15, represents not simply the coefficient of the highest remaining term, but also the coefficient of the corresponding term in the quotient. Further, note that -2 has been multiplied by $3x^2$, by $4x$, and by 15, and the product *with sign changed* has been added to the corresponding term of the dividend.

$$
\begin{array}{r}
x - 2 \\
+2 \overline{)\,3 - 2 + 7 - 5} \\
\quad + 6 + 8 + 30 \\
\overline{3 + 4 + 15 (+ 25}
\end{array}
$$

This represents the division of $3x^3 - 2x^2 + 7x - 5$ by $x - 2$; the -2 has been replaced by $+2$, subtraction has been replaced by change of sign and addition; the remainders have not been written but are merged in the quotient, which is written below.

Divide $5x^4 + 8x + 7$ by $x + 2$.

$$\begin{array}{r|l} -2 & 5 + 0 + 0 + 8 + 7 \\ & \quad -10 + 20 - 40 + 64 \\ \hline & 5 - 10 + 20 - 32 \,(+71 \end{array}$$

Interpreted: $5x^4 + 8x + 7$ divided by $x + 2$ gives as quotient $5x^3 - 10x^2 + 20x - 32$ and 71 as remainder.

The above operations may also be interpreted:

1. $3x^3 - 2x^2 + 7x - 5 = (x - 2)(3x^2 + 4x + 15) + 25.$
2. $5x^4 + 8x + 7 = (x + 2)(5x^3 - 10x^2 + 20x - 32) + 71.$

Substituting in (1) $x = 2$, the right-hand member reduces to the constant term, and the left-hand member is the original expression with 2 substituted for x. Similarly in (2), substituting -2 for x shows that the remainder obtained by dividing by $x + 2$ is equal to the original expression with -2 substituted for x.

If a rational, integral, algebraic expression in x is divided by $x - a$ (a may be negative), the division being continued until the remainder does not contain x, the remainder is equal to the original expression with a put for x. Let us represent by $f(x)$ an expression of this kind, *i.e.* an expression consisting only of the algebraic sum of positive integral powers of x with constants as coefficients.

Then, you can divide $f(x)$ by $x - a$, obtaining a quotient, $Q(x)$, also a function of x, and a remainder which does not contain x; by definition of division, $f(x) = (x - a)Q(x) + R$; substituting a for x, $f(a) = (a - a)Q(a) + R$.

$$f(a) = 0 + R,$$
$$\text{or } R = f(a). \quad \text{Q. E. D.}$$

The common "check by nines" may readily be proved by the remainder theorem:

When 12,738, which may be written

$$1 \times 10^4 + 2 \times 10^3 + 7 \times 10^2 + 3 \times 10 + 8,$$

in powers of 10, is divided by $10-1$, the remainder is equal to the original expression with 1 put for 10. Hence the remainder when 12,738 is divided by 9 is $1+2+7+3+8$, or 21, or 3 (since 21 divided by 9 gives 3 as remainder, or $2+1$).

Thus, $a \times 10^n + b \times 10^{n-1} + c \times 10^{n-2} + \cdots g \times 10 + h$ divided by $10-1$ gives $a+b+c+\cdots+g+h$ as remainder.

Division by **11**, $10+1$, gives as remainder the sum of the odd coefficients less the sum of the even coefficients, counting from units' place; a sum to 11 can, of course, be dropped as it occurs, or 11 can be added to make the remainder a positive number.

Illustrative examples. — Dividing by the method of detached coefficients:

1. Divide $5x^3+3x^2+2x+6$ by $x-2$ and by $x+2$.

$$\begin{array}{r} 2)\overline{5+\ 3+\ 2+\ 6} \\ \underline{+10+26+56} \\ 5+13+28(+62 \end{array} \qquad 5x^3+3x^2+2x+6 = (x-2)(5x^2+13x+28)+62.$$

$$\begin{array}{l} x+2 \\ -2)\overline{5+\ 3+\ 2+\ 6} \\ \underline{-10+14-32} \\ 5-\ 7+16(-26 \end{array} \qquad 5x^3+3x^2+2x+6 = (x+2)(5x^2-7x+16)-26.$$

2. Divide 5326 by 98 and 102; *i.e.* by $100-2$ and $100+2$.

$$\begin{array}{l} 100-2 \quad 54 \\ +2)\overline{5326} \\ \quad\ \underline{10} \\ \quad\ 426 \\ \quad\ \underline{8} \\ \quad\ 34 \end{array}$$

$5326 = 98 \times 54 + 34$.

Note that the 500 is subtracted mentally, and since 5×2 too much has been taken away, 5×2, or 10, is added to the remainder 32; again, 4×2 too much has been taken away, hence, 8 is added to the remainder.

$$\begin{array}{l} 100+2 \quad 52 \\ -2)\overline{5326} \\ \quad\ \underline{-10} \\ \quad\ 226 \\ \quad\ \underline{-4} \\ \quad\ 22 \end{array}$$

$5326 = 102 \times 52 + 22$.

Here, when 500 is taken away, there still remains 5×2 to be subtracted, since the divisor is 102.

3. Divide 127,384 by 96, by 103, by 124.

```
96 __1326          127,384 = 96 × 1326 + 88.
4)127384
   4
  313
  12
  258
   8
  664
  24
   88 rem.
```

A little practice will enable one to perform the additions mentally, writing:

```
96 __1326
4)127384
   31
    25
     66
      88 rem.
```

```
103 __1236                                   103 __1236
- 3)127384          or, abbreviated,         - 3)127384
   - 3                                          24
    243                                         37
   - 6                                           69
    378                                          76 rem.
   - 9
    694
   - 18
     76 rem.
```

```
120 + 4 __1027
  - 4)127384
    - 4
     338
    - 8
     904
    - 28
      36 rem.
```

In this division the remainders when dividing by 120 must be noted mentally; thus, 330 less 240 gives 90, whence 98 as remainder, from which 8 is to be subtracted. Similarly, 900 less 840 gives 60, to which the 4 is added mentally, and 28 subtracted leaves 36 as remainder.

A number of two places ending in 9 or 8 can be used as divisor in short division form. Thus,

2159 with 3 rem.
59)127384

Think 60 in 127, **2**, with remainder 7; add 2 to 7, giving 9; 93 by 60 gives **1**, with remainder 33; 33 + 1 = 34; 348 divided by 60 gives **5**, with remainder 48; 48 + 5 = 53; 534 by 60 is **9** *less 6;* add 9 to − 6 gives + 3.

1447 with 48 rem.
88)127384

Think of 90 − 2 as divisor.

2
393

127 ÷ 90 = **1**, remainder 37; 37 + 2 = 39.

8
418

393 ÷ 90 = **4**, remainder 33; 33 + 8 (since 4 × 2 too much has been subtracted) = 41.

8
664

418 ÷ 90 = **4**, remainder 58; 58 + 8 = 66.

14
48 rem.

664 ÷ 90 = **7**, remainder 34; 34 + 14 = final remainder, 48.

PROBLEMS

1. Divide $2x^3 + 3x^2 - 9x - 5$ by $x - 1$, $x - 2$, $x - 3$, $x + 1$, $x + 2$, and by $x - 3$, employing synthetic division, and interpret results.

$$\begin{array}{r} +2)2+3-9-5 \\ \underline{+4+14+10} \\ 2+7+5(+5 \end{array} \qquad \begin{array}{r} -2)2+3-9-5 \\ \underline{-4+2+14} \\ 2-1-7(+9 \end{array}$$

$2x^3 + 3x^2 - 9x - 5 = (x - 2)(2x^2 + 7x + 5) + 5.$

When $x = 2$, $2x^3 + 3x^2 - 9x - 5 = +5.$

$2x^3 + 3x^2 - 9x - 5 = (x + 2)(2x^2 - x - 7) + 9.$

When $x = -2$, $2x^3 + 3x^2 - 9x - 5 = +5.$

2. Divide $x^2 - 8x^2 + 2x - 5$ by $x - 1$, $x - 2$, and $x - 3$. What is the value of this function of x for $x = 1$, $x = 2$, $x = 3$?

3. Divide $x^4 - 3x^2 - 18$ by $x - 2$ and by $x - 4$.

4. Divide $x^3 - 1{,}289{,}000$ by $x - 100$ and $x - 110$. What is the value of $x^3 - 1{,}289{,}000$ for $x = 100$ and $x = 110$?

5. Divide $x^3 + 8x^2 - 7x - 2$ by $x - 1$ by synthetic division.

6. Divide $2x^3 - 5x^2 + 7x - 10$ by $x - 2$.

7. Divide 38,942 by 96 (as 100 − 4).

8. Divide 38,942 by 59 (as $60 - 1$) by short division.

9. Divide 38,942 by 104 (as $100 + 4$).

10. Divide $3t^4 - 2t^3 - 18t^2 + 6t - 11$ by $t - 1$. Find the quotient and remainder.

11. Find the value of $x^3 - 179.63$ when $x = 5.6$.

12. Find the square root of 321.62 by repeated division and approximation, as indicated.

13. What is the approximate value of the square root of (*a*) 145; (*b*) 147; (*c*) 150; (*d*) 26; (*e*) 2615?

14. Approximate the cube root of 126; 81; 27.4; 64.2; 127; 218; 350; 520; 735; and 1004.

15. Perform the following divisions by synthetic division; give quotient and remainder; time yourself; 20 minutes is an ample allowance.

a. $x^3 - 3x^2 + 7x - 5$ by $x - 2$.
b. $2x^3 + 4x^2 - 7x + 8$ by $x - 3$.
c. $3x^3 - 2x^2 - 8x + 10$ by $x + 2$.
d. $4x^3 - 2x^2 + 8x - 5$ by $x - .5$.
e. $x^4 - 7x^2 + 8x - 5$ by $x - 2$.
f $x^3 - 18{,}700$ by $x - 25$.
g. $2x^3 - 100x^2 - 18{,}700$ by $x - 300$.
h. $x^3 - .2x - .05$ by $x - .8$.
i. $x^5 - 3x^2 - 100$ by $x - 3$.
j. $x^3 - 2{,}000{,}000$ by $x - 140$.

16. Divide by the method corresponding to synthetic division; time yourself; carry the division to four significant figures.

a. 1375 by $100 - 2$ (or 98).
b. 1375 by $60 - 2$.
c. 4356 by $100 + 2$.
d. 8.736 by 5.9.
e. 6248 by 7.9.

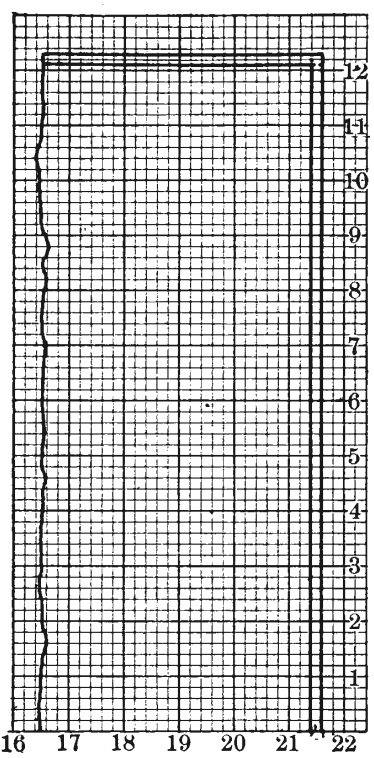
6. Percentage of error. — When any measurement of physical objects is given, the measurement has a certain limit of accuracy, determined in part by the instruments and methods of measurement and in part by the very nature of the thing measured. In measuring the distance to the sun from the earth, at some fixed time, the measurement may be given as 93,000,000 miles ± 1,000,000 miles, or 93,000,000 miles, within a million miles; the thickness of a watch spring may be measured as .014 inch with a possible error of one thousandth of an inch, or .014 ± .001 inch. However, from the point of view of the physicist and mathematician, the distance to the sun is more accurately given than the thickness of the watch spring, for the percentage of error — ratio of possible error to measured value — in the case of the sun's distance is slightly more than 1 % of the distance, while in the other case it is more than 7 % of the thickness of the spring.

Every number which represents a measurement involves this type of error. Obviously, in any computations with such numbers, results are significant only within limits determined by the percentage of error.

7. Significant figures. — The significant figures in any number representing a measurement are those which are given by the measurement, and do not include those initial or terminal zeros which are determined by the unit in which the measurement is made. The terminal zeros in 93,000,000 are not significant figures, as the unit of measurement here is evidently a million miles; as the measurement can be made to one further place, the distance may be written, in "standard form," 9.3×10^7 miles or 9.30×10^7 miles in which only significant digits appear in the first factor combined with powers of 10. In the thickness .014 inch, the initial zero is not a significant figure, as it is apparent that the measurements are made in thousandths of an inch; in "standard form," this is 1.4×10^{-3} inches.

8. Measurement computations. Products. — If the length of a rectangle is measured with an error of less than 1 % of its

true value, and if the breadth is given absolutely, the true area will be given with the same percentage of error as the length. But if the breadth is also only approximately measured, the possible error in the area obtained as the product will be greater than if only one factor involves a possible error. The graph represents the right-hand end of a rectangle whose length and breadth are measured as 21.5 cm. and 12.2 cm. respectively, where it is understood that the measurement only pretends to give these dimensions to within one millimeter, one tenth of one centimeter. The meaning of these figures then is that the length lies between 21.4 cm. and 21.6 cm., and the breadth between 12.1 cm. and 12.3 cm. The first is an error of less than $\frac{1}{2}$ of 1 % and the second of less than 1 %. The uncertainty in area due to the possible error in length is indicated by the areas at the right end with dimensions .1 cm. by 12.1 cm. or .1 cm. by 12.3 cm.; the uncertainty in area due to the breadth measurement is of dimensions .1 cm. by 21.4 cm. or .1 cm. by 21.6 cm. The area uncertainty is then at most .1 cm. by (21.6 + 12.2) cm. or 3.39 sq. cm. of area. This error may evidently affect the third figure in our computation of the area and hence in the product the figures beyond the third place are not significant, and give no real information concerning the actual area in question.



Measurement of an area

A rectangle measured as 21.5 cm. by 12.2 cm.

Note that the area as the product of 12.2 by 21.5 is 262.30

sq. cm., but the inaccuracy of measurement of the length means that there is an uncertainty of area at the right-hand end amounting to $\pm$ 1.22 sq. cm. (cm^2.), and similarly at the top an uncertainty of $\pm$ 2.16 cm^2.; the total uncertainty of area amounts to more than 3 cm^2., and should be given as $\pm$ 3.38 cm^2., or $\pm$ 3.4 cm^2. Commonly, of course, the measurements 12.2 cm. and 21.5 cm. mean that the area has been measured to one half of the last unit given; thus this area actually falls between rectangles of dimensions 12.25 by 21.55 and 12.15 by 21.45; even in this case the area uncertainty is greater, by precisely similar reasoning, than $\pm$ 1.5 cm^2 and is approximately 1.7 cm^2. To give 262.30 as the area of this measured rectangle is giving nonsense in the last two places; it should be given as 262 or 262.3 $\pm$ 1.7 cm^2.

Let k and k' represent measured quantities given with possible errors of i % and e % respectively, e and i being assumed as smaller than unity (in common practice); the absolute values of these measured quantities lie between $k\left(1+\frac{i}{100}\right)$ and $k\left(1-\frac{i}{100}\right)$ and between $k'\left(1+\frac{e}{100}\right)$ and $k'\left(1-\frac{e}{100}\right)$. The true product lies, then, between $kk'\left(1+\frac{i+e}{100}+\frac{ie}{100000}\right)$ and $kk'\left(1-\frac{i+e}{100}+\frac{ie}{10000}\right)$; in other words, the true product may vary by $\frac{i+e}{100}$ from the computed product; $\frac{ie}{10000}$ is disregarded, if i and e are less than 1, since the fraction is less than 1 % of 1 % of kk'. In the graph the product $\frac{ie}{10000}\times kk'$ is represented by one of the small corner squares with dimensions .1 cm. by .1 cm.

Illustrative example. — The product of 987 by 163 wherein each number is correct to within $\frac{1}{2}$ a unit need be computed only to the fourth significant figure as the percentage error may be as great as $\frac{1}{20}$ of 1 % + $\frac{3}{10}$ of 1 %, since $\frac{1}{2}$ in 987 parts is approximately $\frac{1}{2000}$ or $\frac{1}{20}$ of 1 %, and $\frac{1}{2}$ in

163 is greater than $\frac{3}{1000}$ or $\frac{3}{10}$ of 1 %. The error in the product may be as great as $(\frac{1}{20} + \frac{3}{10})$ of 1 % or $\frac{7}{20}$ of 1 % ; but $\frac{7}{20}$ of 1 % of any number certainly affects the fourth place and probably affects the third place in the number. Hence there is no point whatever in carrying this computation beyond four places,

987	987
163	163
98700	987 begin with 100 × 987.
59220	592 take 6 × 98, carrying however the 4 from 6 × 7.
2961	29 take 3 × 9, carrying the 2 of 3 × 8.
160881	160800

163	$987 + \frac{1}{2}$	$987 - \frac{1}{2}$
987	$163 + \frac{1}{2}$	$163 - \frac{1}{2}$
1467	$160881 + \frac{1}{2}(987 + 163) + \frac{1}{4}$	$160,881 - \frac{1}{2}(987 + 163) + \frac{1}{4}$
130	$- 161456\frac{1}{4}$	$- 160306\frac{1}{4}$
11		
160800 *ans.*		

The product of 987 × 163 is 160,881 ; $987\frac{1}{2} \times 163\frac{1}{2}$ gives $161,456\frac{1}{4}$; $986\frac{1}{2} \times 162\frac{1}{2}$ gives $160,306\frac{1}{4}$; the actual area, if these represent dimensions of a rectangle measured to three significant figures lies between $160,306\frac{1}{4}$ and $161,456\frac{1}{4}$. In practice we take the product 987 × 163 to four significant figures, which gives the area slightly more accurately than our measurements justify.

9. Abbreviated multiplication of decimals. — The abbreviated process of multiplication applies particularly well to decimal fractions, but the method can be extended to integers quite as well. To find .9873 × .1346 correct to four decimal places.

.9873	
.1846	
9873	begin with the highest digit of the multiplier ; first × fourth decimal place gives fifth decimal place.
7898	continue with 8 × 7 (second × third place), carrying the 2 from 8 × 3.
395	take 4 × 8 (third × second place) carrying 3 from 4 × 7, or 28, which is more than 2 units in the fifth place.
59	6 × 9, or 54, + 5 carried from the 6 × 8.
18225	It assists in the process to cross out the last upper digit as it is used ; thus here 3 would be crossed out first, then 7, then 8, and finally 9.

If a check is desired, multiply again, reversing the order of the factors; thus:

.1846	
.9873	
16614	begin with 9×6.
1477	take 8×4, adding 5 from the 8×6 product.
129	take 7×8, adding 3 from the 7×4 product.
5	take 3×1, adding 2 from the 3×8 product.
.18225	Read this as .1823 to four places.

Obviously, if these were integers, you could proceed in the same way, writing the final product with four zeros, as 18,230,000.

A similar abbreviation can be effected in division by dropping each time the last figure of the divisor used, and using the remaining part of the original divisor as new divisor.

Thus, to divide .18225 by .9873 or by .1846 you proceed as follows:

.1846	.9873
9873).18225	1846).18225
9873	16614
8352	1611
7898	1477
454	134
395	129
59	5
59	5

Here 9873 is used as the first divisor; then 987 is used, but to the partial product, 8×987, is added the tens' digit of 8×3, the digit just crossed out; then 98 is taken as divisor and to the product is added the tens' digit of 4×7 (28 is taken as giving a tens' digit of 3); then 9 is used and 5 carried over from 6×8.

10. Percentage effect of errors in divisor. — If a divisor is known to be too large or too small by a definite percentage of itself, the quotient will be respectively smaller or larger than the correct quotient, for small per cents, by approximately the same per cent.

By division, $\frac{1}{1+i} = 1 - i + i^2 - i^3 + i^4 - \cdots$

$$\frac{1}{1-i} = 1 + i + i^2 + i^3 + i^4 + \cdots.$$

For values of i less than .05, i^2 is less than 0025, or $\frac{1}{4}$ of 1 %; i^3, i^4, and i^5 are less than .000125, .00000625, and .0000003125, respectively. Hence an error of from 1 % to 5 % of excess in the divisor means an error of deficiency varying also from 1 % to 5 %, within $\frac{1}{4}$ of 1 %, or from .99 % to 4.75 %, or from .9901 % to 4.7625 % in the quotient. For values of i between $\frac{1}{2}$ of 1 % and 1 %, an error of deficiency in the divisor means the same error of excess in the quotient, within $\frac{1}{100}$ of this error. The meaning in physical measurements of these results is that when the divisor is correct only to the third significant figure, with a possible error of $\frac{1}{2}$ to 1 unit in the third place, the quotient will be correct to about the same degree of accuracy.

For three-place numbers the divisor may vary from 100 to 999. The possible error of $\frac{1}{2}$ unit, $\pm \frac{1}{2}$, means that 100 must be replaced by $(100 \pm .5)$ or $100(1 \pm .005)$ and 999 by $999 \pm .5$, or approximately $999(1 + .0005)$; the quotient will vary from $1 - .005$ to $1 - .0005$ times the obtained quotient. Hence the quotient obtained is valuable at most to the fourth place, and frequently not beyond the third place.

Illustration. — Given that 76,430 is divided by 180; what variation in the quotient would a change of 1 in the divisor produce?

$76{,}430 \div 180 = 424.6.$

Suppose that instead of 180, 179 should have been used. What is the error in the quotient? $180 - 1 = 180(1 - .006)$, the error in the divisor is more than .5 % and less than .6 % of the divisor; the error in the quotient is no more than 2.5 and no less than 2.1, since 1 % of 424.6 is 4.246 and .6 % and .5 % are respectively 2.5 and 2.1; the quotient may be taken as 427.1, whereas 426.9 is obtained by actual division. Even an error of $\frac{1}{2}$ a unit in the divisor 180 affects the third place in the quotient. In obtaining .5 % and .6 % of 424.6, there is no point in carrying the work beyond two places; the values show that the error is between 2.1

and 2.5, and further places add nothing to the accuracy. The fraction $\frac{1}{100}$, or in the case of the $\frac{1}{2}$ unit error of $\frac{1}{3600}$, might just as well be used as per cents. This gives in the latter case $\frac{1}{3600}$ of 424.6, or + 1.2 as correction, giving 425.8 as quotient; the actual quotient is 425.794.

Do not carry divisions and multiplications beyond the degree of accuracy warranted by the data.

Illustrative examples. — You can multiply 3.14159 by 140.8 and obtain the result numerically correct to six decimal places. But if the 140.8 represents the diameter of a circle, measured correctly to the tenth of an inch (or of a foot, or of a meter) the product of 140.8 by 3.14159 gives a valuable result *only to the first decimal place;* the circumference cannot be computed correctly to any further percentage of accuracy than that with which the diameter is measured. The area can be computed here with any meaning only to four significant figures; in fact an error of $\pm .05$ inch in the diameter makes a possible error of ± 10 square inches in the area. It is convenient to write the products from left to right, dropping work beyond the second decimal place.

140.8	140.8	
$3\frac{1}{7}$	3.14159	the .00059 is of no use as it
422.4	422.4	does not figure in the product.
20.1	14.1	
442.5 circumference	5.6	
	1	
	442.2 circumference	

For most practical purposes $3\frac{1}{7}$ is sufficiently accurate, as in finding this area, πr^2:

70.4
70.4
4928
28
4956 only 4 places to be retained.

4956	4956
$3\frac{1}{7}$	3.14159
14868	14868
708	496
15576 area.	198
	5
	15567 area.

Area as found to correspond to data, **15570.**

PROBLEMS

1. The distance of the earth from the sun varies between 91.4×10^6 miles and 94.4×10^6 miles. The length of the earth's orbit lies between circles having these lengths as radii. Between what values does this orbit lie? What is the approximate orbital speed of the earth in miles per hour?

2. The mean distance of the earth from the sun is 92.9×10^6 miles. Compute the circumference and the mean speed and compare by percentages with the preceding.

3. Compute the speed of a point on the earth due to the rotation, taking that at latitude 45° the radius of the circle of latitude is 3050 miles. Compare the rotational speed with the revolutional speed.

4. What effect on the computed velocity would it have to take 365.25 instead of 365? How would you correct your division for 365.25 as divisor after having obtained the quotient, dividing by 365? what change would using 365.26 instead of 365.25 effect in the computed velocity?

5. The distance of the moon from the earth varies between 221,000 and 260,000 miles, mean 238,000; discuss the length of the path of the moon and the velocity of the moon which has a periodic time of 27.32 days.

6. A man whose salary is $3000 pays $480 for rent. What per cent is this of his salary? Suppose that he earns $275 in addition to his salary, what per cent is the rent paid of his income? Compute only to tenths of one per cent.

7. If a man with an income $3275 pays $1100 per annum for food, $630 for clothes, $240 for life insurance, $200 for "higher life," and saves the balance, compute his budget by per cents.

8. Given that a pendulum of length l cm. makes one beat, one oscillation, in t seconds, connected by the relation,

$$t = \pi\sqrt{\frac{l}{980}},$$

find the length l to two decimal places when $t = 1$.

9. What effect on l does a change from 980 to 981 produce? What decimal place in l would be affected?

10. What error would the use of $3\frac{1}{7}$ instead of 3.14159 introduce?

11. The number n of vibrations of a pendulum of length 99.39 cm. is 86400, when $g = 980.96$; g is the acceleration due to gravity, and the formula for the number of vibrations is given by the formula,

$$n = \frac{86400}{\pi}\sqrt{\frac{g}{l}},$$

or for the seconds' pendulum, when $l = 99.39$, $g = 980.96$, it is $n = 86{,}400$. Suppose that at the top of a mountain (g diminishes) this pendulum of length 99.39 loses 86 beats per day, what is the approximate percentage of loss in n? The percentage of loss in g is approximately double this since $\sqrt{1-i} = 1 - \frac{i}{2} - \frac{i^2}{8}$. What is the approximate loss in g? Take 86 as $\frac{1}{10}$ of 1 % of 86,400.

12. Given $g = 980$, $l = 50$, compute n in the formula of problem 11. What maximum effect on n would a change from $l = 50$ to $l = 50.5$ cm. produce?

13. Compute the weight of a table top, hardwood, dimensions correct to .05 foot, top $48.1 \times 36.4 \times 2.1$ inches. Weight of wood 48 pounds per cubic foot.

14. If a table top similar to the above weighs 97 pounds, compute the weight per cubic foot of the wood.

15. If water weighs 62.4 pounds per cubic foot, compute the specific gravity of each of the preceding woods.

$$S = \frac{\text{wt. of cubic foot of wood}}{\text{wt. of cubic foot of water}}.$$

16. The path of the earth is approximately a circle of radius 92.9×10^6 miles, of which the center is approximately $1\frac{1}{2}$ million miles from the sun. Compute this circumference and compare with the results in problems 1 and 2.

CHAPTER III

EXPONENTS AND LOGARITHMS

1. Exponent laws. — For convenience the product of a by itself, $a \times a$, is represented by a^2, $a \times a \times a$ by a^3, ..., and $a \cdot a \cdot a \cdot a$ to m factors by a^m. In this notation m is called the exponent and a the base. The following laws evidently hold:

$$\text{I.} \quad a^m \cdot a^n = a^{m+n}.$$

$$\text{II.} \quad \frac{a^m}{a^n} = a^{m-n}, \text{ when } m > n.$$

$$\text{III.} \quad (a^m)^n = a^{m \cdot n}$$

$$\text{IV.} \quad (a \cdot b)^m = a^m \cdot b^m.$$

In the definition as given, m represents the number of factors and is assumed to be a positive integer. However, it is found possible to define a^m for all real values (fractional, negative, zero, irrational) of m so as to have the resulting numbers combine according to the four laws given above.

Thus, $a^0 \cdot a^m = a^{0+m} = a^m$, if Law I is to continue to hold; hence, a^0 must be defined to equal 1, since multiplying a number, a^m, it gives that number. To be justified in using a zero exponent with this meaning the other exponent laws must be shown to hold when either m or n is zero, but in II only n could be zero at this point.

For a negative integer, $-n$, if Law I is to hold, a must be defined as such a number that $a^n \cdot a^{-n} = a^{-n+n} = a^0 = 1$; hence we define a^{-n} as the reciprocal of a^n, $a^{-n} = \frac{1}{a^n}$. All the laws I to IV can be shown to hold under this extension of the meaning of a^n.

Similarly, $a^{\frac{p}{q}}$, if Law III is to hold, must represent a number which raised to the qth power equals a^p; $a^{\frac{p}{q}}$ is thus defined as the qth root of the pth power of a. Taking this definition of $a^{\frac{p}{q}}$, Laws I to IV can be shown to hold with this extension in possible values of m and n; p and q are assumed to be integers.

For fractional exponents a fifth law is introduced, $a^{\frac{p}{q}} = a^{\frac{mp}{mq}}$.

For irrational values of n, a^n is defined by a limiting process. Thus, $a^{\sqrt{2}}$ is defined as the limit of the series $a^{1.4}$, $a^{1.41}$, $a^{1.414}$, ..., wherein the successive rational exponents define the square root of 2.

The operations of elementary algebra with radicals are made subject to the exponent laws.

Thus, $2\sqrt{3} = 2 \cdot 3^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} = (2^2 \cdot 3)^{\frac{1}{2}} = 12^{\frac{1}{2}} = \sqrt{12}$,
by successive application of V, III, and IV.

The operation of raising to a power indicated by a^m, with m integral, is called involution. The inverse operation of finding x when x^m is given equal to a is called evolution.

2. Logarithms. — *A logarithm is an exponent.*

The relation $x = a^m$ may be written $m = \log_a x$.

m is the exponent which applied to ***a*** gives ***x***;

m is the logarithm of ***x*** to the base ***a***.

3. Fundamental laws of logarithms.

a. **Logarithm of a product.**

If $x = a^m$ and $y = a^n$,

$$x \cdot y = a^m \cdot a^n = a^{m+n}$$

$$\log_a (x \cdot y) = m + n = \log_a x + \log_a y.$$

In the language of logarithms and translated into ordinary language this theorem is as follows:

$$\text{I.}\quad \log_a (x \cdot y) = \log_a x + \log_a y;$$

in words, *the logarithm of a product is the sum of the logarithms of the factors.*

b. **Logarithm of a quotient.**

$$\frac{x}{y} = \frac{a^m}{a^n} = a^{m-n},\quad \log_a \frac{x}{y} = m - n = \log_a x - \log_a y.$$

$$\text{II.}\quad \log_a \frac{x}{y} = \log_a x - \log_a y;$$

in words, *the logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.*

c. **Logarithm of a power.**

If $x = a^m$, $x^n = (a^m)^n = a^{mn}$; $\log_a x^n = m \cdot n = n \log_a x$.

$$\text{III.}\quad \log_a x^n = n \log_a x;$$

the logarithm of a power of a number is the index of the power times the logarithm of the number.

Since our exponent laws hold for all values of m and n, these theorems hold for all values of m and n.

x and y are assumed to be positive numbers and for computation purposes 10 is commonly taken as the base.

We assume that as the logarithm increases the number increases. This can be readily proved from the fact that $10^m \cdot 10^n = 10^{m+n}$; no matter how small the n is, as a positive quantity, 10^n is greater than 1. For n any positive fraction, $\frac{p}{q}$, 10^n represents the qth root of the pth power of 10, wherein p and q are integers. Now 10^p will be an integer 10, or greater than 10, and the qth root of this integer will be a number greater than 1 for it will be a number which raised to the qth power equals 10, 100, 1000, or some greater number. It could not be less than 1, for every positive integral power of a number less than 1 is also less than 1.

Exponential Form		Column of Numbers	Column of Numbers :	Column of Logarithms
. . . . .				
10^{-12}	=	000000000001	log .000000000001	= − 12
. . . . .				
10^{-5}	=	00001	log .00001	= − 5
10^{-4}	=	0001	log .0001	= − 4
10^{-3}	=	001	log .001	= − 3
10^{-2}	=	01	log .01	= − 2
10^{-1}	=	1	log .1	= − 1
10^{0}	=	1	log 1	= 0
10^{1}	=	10	log 10	= 1
10^{2}	=	100	log 100	= 2
10^{3}	=	1000	log 1000	= 3
10^{4}	=	10000	log 10000	= 4
10^{5}	=	100000	log 100000	= 5
. . . . .				
10^{12}	=	1000000000000	log 1000000000000	− 12
. . . . .				

These positive numbers, middle columns, are arranged vertically in order of magnitude; the exponents (left) or logarithms (right) also are arranged vertically in order, increasing from − 12 (or from − ∞ indicated by dots above) to − 5, to − 4, to − 3, ··· to 0, to 1, ··· to 12, ··· to ∞. As the number increases the logarithm increases. Placing any number, not an integral power of 10, in its proper place as to magnitude on such a diagram, the logarithm has for integral part the logarithm of the preceding number in the table. Thus, 75.64 falls between 10 and 100 and its logarithm will be 1^{+}; .07564 falls between .01 and .1 and its logarithm will be -2^{+}, meaning − 2 plus some positive fraction.

4. Logarithms, characteristic and mantissa. — Any two-place number lies between 10 and 100; the logarithm will lie between 1 and 2. Any four-place number lies between 1000 and 10,000; the logarithm will lie between 3 and 4, since as the number increases the logarithm increases. The fraction .07 is greater than .01 and less than .1; the logarithm is then greater than -2 and less than -1.

The logarithm of any number between 1 and 10 is a fraction, expressed commonly as a decimal between 0 and 1.

The logarithm of any given number which is expressed in decimal notation can be expressed as an integer, positive or negative, called the *characteristic*, plus the positive decimal fraction, the *mantissa*, which is the logarithm of that number between 1 and 10, having the same succession of digits as the given number. Initial zeros are not included in the succession of digits.

$$\begin{aligned}\log 10^n k &= \log 10^n + \log k \\ &= n \log 10 + \log k \\ &= n + \log k.\end{aligned}$$

Thus, $\log 3.16229 = .50000$, since 3.16229 is the approximate square root of 10.

$$\begin{aligned}\log 31622.9 &= \log (10)^4(3.16229) = \log 10^4 + \log 3.16229 \\ &= 4 + \log 3.16229 \\ &= 4.50000.\end{aligned}$$

$$\begin{aligned}\log .000316229 &= \log (10)^{-4}(3.16229) = \log (10)^{-4} + \log 3.16229 \\ &= -4 + .50000 \\ &= \bar{4}.50000, \text{ only the 4 is negative} \\ &= 6.50000 - 10, \text{ since } -4 \text{ equals } 6 - 10 \\ &- 16.50000 - 20 \\ &- 26.50000 - 30, \\ &\cdots\cdots\end{aligned}$$

The minus sign over the 4 indicates that only the characteristic is negative; the alternate forms for writing a negative characteristic are frequently found convenient to use, partienlarly in extracting roots.

RULE. — *The logarithm of any number greater than unity has as characteristic a positive integer* (0 *included*) *which is* 1 *less than the number of digits to the left of the decimal point.*

The logarithm of any decimal fraction has as characteristic the negative of a positive integer (0 *not included*) *which is* 1 *greater than the number of zeros between the decimal point and the first significant digit* (i.e. *digit other than* 0) *to the right of the decimal point.*

5. Computation of logarithms. — Logarithms are actually calculated by a series derived from formulas obtained in higher mathematics. However, a simple although laborious method of computing logarithms approximately may make the significance of the logarithm somewhat clearer.

$$2^{10} = 1024, \qquad 2^{20} = 1{,}048{,}576.$$

Evidently, $10^6 < 2^{20} < 10^7$, since 1,048,576 is greater than 1,000,000 and less than 10,000,000.

Extracting the twentieth root,

$$10^{\frac{6}{20}} < 2 < 10^{\frac{7}{20}},$$

or

$$10^{.30} < 2 < 10^{.35}.$$

Hence, 2 is greater than 10 with an exponent 30, and less than 2 with an exponent .35.

$2^{40} = (1{,}048{,}576)^2 < 1{,}100{,}000{,}000{,}000$ and greater than $1{,}000{,}000{,}000{,}000$; $2^{80} < 1.21 \times 10^{24}$ (2^{40} by 2^{40}) and greater than 1×10^{24}. Whence $2^{100} < 1.33 \times 10^{30}$ and greater than 1×10^{30}, whence $10^{30} < 2^{100} < 10^{31}$, whence

$$10^{.30} < 2 < 10^{.31}, \text{ or } \log_{10} 2 = .30^{+}.$$

Similarly, 3^{20} is 3,486,783,861, or greater than 10^9 and less than 10^{10}; hence, log 3 lies between .45 and .50, the computation of 3^{20} is easily made by using $3^4 = 81$ and $3^{20} = (81)^5$, multiplying each time by 81 instead of by 3; the partial product by the 1 does not need to be rewritten.