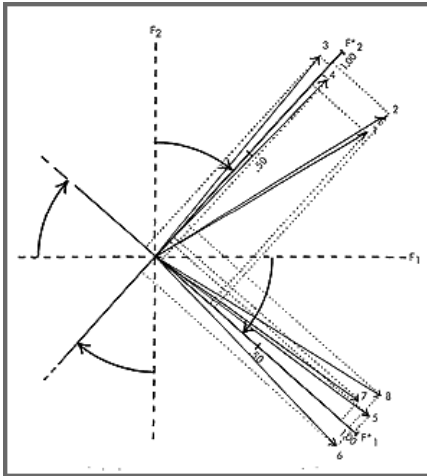


Vector Analysis

An Introduction to Vector-Methods and Their Various Applications to Physics and Mathematics



By

Joseph George Coffin

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VECTOR ANALYSIS

AN INTRODUCTION
TO
VECTOR-METHODS
AND THEIR VARIOUS APPLICATIONS
TO
PHYSICS AND MATHEMATICS

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PREFACE

EVER since the development of Quaternion analysis, by Sir William Rowan Hamilton, and of the "Ausdehnungslehre," by Grassmann, there has been a growing feeling that the older and more common processes of analysis were in some way artificial and complex.

This fact exists, for it is such, because these newer methods and ideas apply more naturally, more simply and more directly to many of the conceptions of geometry, mechanics and mathematical physics, than those long accepted.

Why then have these admitted advantages not led to a more universal adoption of these methods? The answer seems to be that the required change of ideas, of manner of thought and of notation, was too radical. It is well known that changes evolve slowly, and although to many, evolution is far too slow a process, the only way to proceed is to aid to the best of one's ability in bringing about the desired result.

One who has studied and labored over the applications of mathematical analysis to physical and geometrical problems, naturally has reluctance to discard the old familiar looking formulæ and start anew in an unknown and radically different language.

However great the skill and ingenuity shown by the pioneer in solving problems by Quaternions, there was always left the thought to the unbiased student that a lack of parallelism existed between the old and the new methods of treatment. Such a lack undoubtedly does exist, but it is only during the last few years that a method has been

evolved which avoids this fatal defect. It is chiefly through the labors of Gibbs and Heaviside that an analysis has been perfected which not only does away with the *unnecessary* complexity and artificiality of other analyses but offers a strictly natural and therefore as direct and simple a substitute as possible, and, at the same time in no wise is at variance, but runs parallel to them.

This new, yet old method is Vector Analysis; it combines within itself most of the advantages of both Quaternions and of Cartesian Analysis.

The adoption of Vector Analysis is urged on the grounds of naturalness, simplicity and directness; with it the true meaning of processes and results is brought out as clearly as possible, and desirable abbreviation is obtained.

It is admitted, that to a straight and clear thinker, almost any notation or mathematical method suffices, and to such a one, changes in notation or method may appear hardly worth while. He has already attained one of the results which, perforce, follow the intelligent assimilation of a vector method of thinking. To him there is left but the attainment of a simple notation which is the logical accompaniment of clear thought. A few examples of vector concentration are to be found in the exercises of the last chapter of this book. But the sole use of vector notation, without the insight and clear conceptions which should obtain at the same time, is without any value whatsoever, vitiates the vector point of view, and is contrary to the spirit of it.

It is almost unnecessary to state that the mind of the physicist ought to be of the visual type so well exemplified in the mind of Faraday. He should see the lines of force emerging from the magnet; see that they are continuous within the metal; follow them, in his mind's eye, as they are displaced by various causes; he should have some sort of a visual conception of the manner in which the electro-magnetic waves are traveling through the ether

around him; to him the divergence and the divergence theorem should have a simple meaning.

To a mind other than this, the study of mathematical physics must be merely a series of analytical transformations without the vitality of their visual significance. To purely analytical minds, as distinct from the visual or intuitive type, the methods of Vector Analysis reduce to little more than an analytical shorthand. To the intuitive mind, however, they are illuminative and simplifying, allowing the mind to grasp and the hand to write the essential facts and transformations, unembarrassed with the generally undesirable complexity of Cartesian symbolism. It is impossible to study and to apply Vector Analysis to problems and not to have one's ideas and thought made clearer and better by the labor involved.

There are very good reasons for all these advantages. In Nature we are confronted with quantities called scalars which have size or magnitude only, and also with other quantities called vectors which have direction as well as magnitude. In order to manipulate vector quantities by the older methods, they were decomposed into three components along three arbitrary axes and the operations made upon these components. Is it not evident that the bringing in of three arbitrary axes is an artificial process, and that the decomposition of the vector into components along these axes is also artificial, unnatural even? Why not go directly to the vector itself and manipulate it without axes and without components? To do this, is possible, and in the following pages an attempt is made to show how it may be done.

There is still another ground for urging more extended study of Vector Analysis than now obtains. So many physicists of renown have been converted to its methods and use that to ignore their leadership is an impossibility. When such men as Lorentz, Foppl, Heaviside, Bucherer, Gibbs,

Abraham, Bjerknes, Sommerfeld, Cohn and many others are converted to its use, it is high time that the student familiarize himself at least with vector notation, even if not to become an expert in its use.

No one can deny the vast improvement that has taken place, in recent years, in our conceptions of physical processes; and few will deny that a large part of this improvement has been due to the ideas introduced with the advent of vector methods of thought.

That Lagrange reduced all of mechanics to a purely analytical basis without, as he boasts, necessitating diagrams, is certainly a wonderful accomplishment. Yet how much clearer and more elegant if the equations become alive with meaning, if to the algebraic transformations a mental picture of what is taking place is obtained!

Maxwell gave a splendid reference in favor of the new methods when he said, in speaking about the motion of the top, "Poinsot has brought the subject under the power of a more searching analysis than that of the calculus, in which ideas take the place of symbols and intelligible propositions supersede equations."

Vector Analysis has the advantages of Lagrange's analytical method as well as those of the idealogical method of Poinsot.

The writer does not, in any way, urge the rejection of anything of value in any method whatsoever. It is not well nor is it intended that the methods of Vector Analysis should be essentially different from those to which the student is supposed to be accustomed. In fact, it has been the aim throughout this book to evolve an analysis to which all the knowledge of the reader can be immediately applied. and to so expound this analysis, that Cartesian equations may be immediately written in vector notation and conversely.

There is still another important advantage, which should not be overlooked, that is, vector notation just as vector

thought, is entirely independent of any choice of axes, or planes of reference, and yet the transformation of the vector equations into other systems, requiring these axes or planes is always extremely easy. To prove that a *natural* invariant is invariant to a change of axes, has always appeared to the writer an extremely foolish operation and a waste of time. This is not saying, that in a mathematical theory of invariants such a property of an algebraic expression is not instructive or interesting. But to say, for example, that the properties of the lines of force which cut a set of equipotential surfaces at right angles, (*i.e.*, the lines $\mathbf{F} = -\nabla V$) may be dependent upon the particular set of axes used to investigate them, is a waste of time to say the least. How can a truth vary with the language used in expressing it?

No attempt at mathematical rigor is made. Such refinements serve only to conceal the simplicity of fact, which it is the aim of these pages to elucidate. The appearance of extended proofs, the writer considers to be entirely out of place in a book of this kind. On the other hand, no one is more in favor of mathematical rigor than he; the point is simply to eliminate discussions whose presence would lead the attention astray from the main ideas of the argument. In any case, whenever a demonstration does not satisfy the fastidious, the results may be found more rigorously, if not more clearly, established in works devoted to mathematically rigorous demonstrations.

The student will find with a little study that he may easily take down lectures, given in Cartesian notation, directly into vector notation. Serious trial will convince him that time is gained and what is still more important, that equations will be, *must* be, understood if this is done. It is by precisely such a process that the writer familiarized himself with the subject.

The notation adopted is that of Prof. Willard Gibbs, one of the too few great American physicists and mathema-

ticians. The reasons leading to this choice are fully set forth in the Appendix.

The first part of the book is devoted to a concise treatment of the fundamental principles of the subject, the remaining chapters, to the application of the analysis to the *beginnings* of mathematical physics, including geometry, mechanics, magnetism, electricity, heat and hydrodynamics. It was found necessary to omit many beautiful applications in elasticity, electron theory and other parts of physics in order to keep the size of the volume within bounds.

The student who takes up the later chapters, is supposed to be familiar, to a certain extent, with the subjects therein contained, and these chapters are intended to show the beginner how to translate and demonstrate the theorems into the new calculus. The writer therefore makes this his apology for a certain necessary lack of logical sequence in the treatment of the various subjects.

The treatment of alternating currents and allied subjects has been omitted, because in practically every modern book on the subject the notation of the special vector method employed, is fully explained in some part of it.

It is hoped that but few errors still remain in the text. The author, alone, corrected the proof, but numerous equations and special difficulties met with in printing in a new notation, rendered the corrections very difficult and laborious.

The copy has been read by Prof. Saurel, professor of mathematics in the College of the City of New York, and the author wishes to acknowledge here his indebtedness for the kindness as well as for many valuable suggestions.

A detailed list of works on Quaternions is rendered unnecessary by Professor Macfarlane's "Bibliography" published by the "Association for the Promotion of the Study of Quaternions and Allied Mathematics," Dublin, 1904, but a list of works which have been especially consulted is appended to

the preface, and the writer here acknowledges his obligations to all of them. If this book succeeds in making plain the author's particular point of view; in simplifying ideas, or in causing simple ideas to seem clearer than before, he will feel amply repaid for any pains taken in producing what was to him a labor of love.

J. G. COFFIN.

NEW YORK, April 9, 1909.

PREFACE TO SECOND EDITION

IN this new edition a number of small errors which are peculiarly difficult of discovery in a work involving so many different kinds of type have been corrected. The sincere thanks of the writer are due to the large number of correspondents who have greatly helped him in this revision.

The author is glad to be able to state that to his knowledge but one theoretical error has been discovered up to the present time.

Certain portions have been rewritten and fourteen pages of notes have been added to the appendix.

In particular a short digression on different varieties of vectors; certain additional definitions of differential geometry with reference to curves in space which seemed interesting and useful; the demonstration of Frenet's valuable formulæ for space curves; an interesting example of vector reasoning as applied to the solution of the differential equation of motion of an electron in a magnetic field; two new proofs of Stokes' Theorem not found as far as we know in any treatise of vector analysis; an additional proof of Gauss's Theorem; and proofs of two theorems in integration analogous to the Divergence Theorem.

Both the publisher and the writer are delighted with the reception accorded this little book in this country and abroad.

The writer is of the opinion that a great many results of mathematical physics are elementary and easily understood by the student if explained in the right way, and the student thereby finds himself in a position to go right ahead in the more difficult extensions, when he comes to them. This book was written with that end in view. It is practically an elementary course in mathematical physics.

He also hopes that not only will this volume help the student to an acquisition of the fundamentals of Vector Analysis, but that also, and not least, it will awaken in him a desire for further study in that most beautiful and extensive of all branches of study, — Mathematical Physics.

He believes that in this country there is a wealth of material for the making of brilliant investigators in this line, if they are encouraged to approach the higher branches without the fear that it is beyond their capabilities.

He therefore makes a plea for the encouragement of students having ability in this direction so that soon it can no longer be said that we are not up to the standard of the investigators of the Old World. True, they had a long start and we have been handicapped, but we hope in the course of a few years to be abreast of them.

J. G. COFFIN.

NEW YORK, *June*, 1911.

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$$\nabla \equiv \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}.$$

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SUGGESTIONS FOR WRITING VECTOR ANALYSIS ON THE BOARD

A number of inquiries have come in asking how to write vectors on the blackboard. It seems that the bold-faced type or **Clarendon** is perfectly satisfactory as far as print is concerned, but it is impracticable to produce such a difference in chalk-written symbols. To a great extent these same troubles also occur in manuscript.

There are several methods of differentiating vectors from purely scalar symbols which have proved satisfactory.

The notation given in the text is entirely practicable and definite. That is, if a , b or r denote vectors in any discussion, let a_0 , b_0 or r_0 denote their magnitudes and a_1 , b_1 or r_1 denote their directions or unit vectors along a , b or r respectively.

Thus

$$a = a_0 a_1.$$

There is here a slight chance of ambiguity in the equation

$$a = a_1 i + a_2 j + a_3 k$$

where the i -component of a might be confounded with the unit vector along a . The writer does not consider this a serious objection. Like Tait we say that anybody finding difficulty with this small matter has begun the study of vectors too soon!

Another method is to place a line or dash over the vector-symbol. So that if a denotes a vector, then a is its magnitude, $\bar{a}_1$ is its direction and a_1 is its i -component.

Still another method to which the writer is very partial, having been brought up on Hamilton's notation, is to reserve the Greek alphabet for vectors.

So that if α is any vector, α_0 or a is its magnitude, and α_1 is its direction, while a_1 , a_2 , a_3 are its i -, j -, k -components.

After reading the book notices and reviews we are still of the same opinion as to the essential superiority of Gibb's notation over others, notwithstanding the criticisms of it which we expected. We have never claimed that $a \times b$ was a symmetrical *function*, but we do claim that the *notation* of both $a \cdot b$ and $a \times b$ is symmetrical. The minus sign in $a \times b = -b \times a$, does not make the *notation* unsymmetrical.

Almost simultaneously with this text was issued a vector analysis by the Italian mathematicians Burali-Forti and Marcolongo.

These gentlemen have invented still another notation which is similar to ours but which employs the $\times$ (large cross) for a scalar product and an inverted V (Λ) for a vector product. With the symmetry of their notation we are in favor, but why introduce any more notations when there are already so many to pick from?

This question of notation, which has nothing to do with the spirit of the method, is for each individual to solve for himself. We have employed what *we* believe to be the simplest and best and we have presented at length our arguments in favor of it.

J. G. COFFIN, 1911

VECTOR ANALYSIS

CHAPTER I.

ELEMENTARY OPERATIONS OF VECTOR ANALYSIS.

Definitions.

1. A **Vector** is a directed segment of a straight line on which are distinguished an initial and a terminal point. A vector thus has a magnitude and a direction. Any quantity which can be represented by such a segment may be called a vector quantity. The importance of this generalized conception is easily understood when it is considered that motion or displacement, velocity, acceleration, force, electric current, magnetic flux, lines of force, stresses and strains due to any cause, flow of heat and of fluids, all involve two parts, *i.e.*, magnitude and direction. All such quantities are vector quantities.

A **Scalar** on the other hand is any quantity which although having magnitude does not involve direction. For example, mass, density, temperature, energy, quantity of heat, electric charge, potential, ocean depths, rainfall, numerical statistics such as birth rates, mortality or population, are all scalar quantities.

A scalar, then, reduced to its simplest terms is merely a number and as such obeys all the laws of ordinary algebraic analysis. A vector, however, involving direction in addition to its numerical magnitude has an analysis peculiar to itself, the laws of which are to be derived.

2. **Graphical Representation of a Vector.** Any vector quantity may be represented graphically by an arrow.

The tail of the arrow, O , is called the origin; the head, A , is called the end or terminus.

Symbolically a vector may be denoted by two letters, the first one indicating the origin, the second one the terminus.

A small arrow is often placed over these letters to indicate more exactly that the quantity considered is a vector.

Thus, OA denotes the vector beginning at O , ending at A , and pointing in the direction from O to A . This notation while useful is at times cumbersome. Hence more usually a vector will be denoted by a single letter, which involving more than a mere scalar is printed differently to distinguish it from purely scalar quantities, *i.e.*, in **Bold-faced Type**.

Thus the vector $\mathbf{a}^*$ means the going of the distance OA in the direction O to A from *any point in space as origin*.

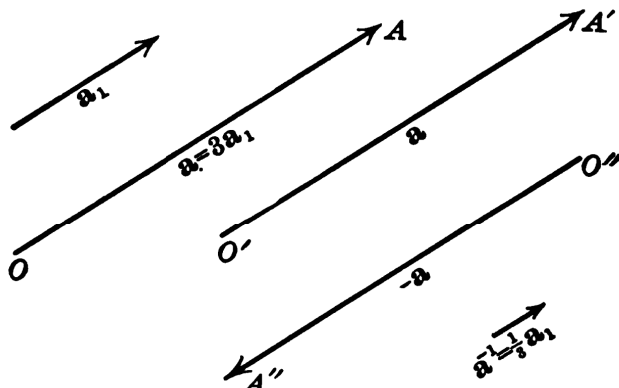


FIG. 1.

3. Equality of Vectors. All lines having the same length or magnitude and the same **sense** are equal vectors whatever their origin may be. Thus in Fig. 1, OA and $O'A'$ are equal vectors.

Negative Vector. The vector $O''A''$ having the same length and direction as $\mathbf{a}$ but the opposite sense, is defined

The terms *Step*, *Stroke*, or *Directed Magnitude* are sometimes used *Anonyms of Vector*.

as the negative of $\mathbf{a}$ and is written $-\mathbf{a}$. Evidently also OA is the negative of $O''A''$.

Unit Vector. The directional part of any vector $\mathbf{a}$ may be concisely represented by a vector having the same sense and direction as $\mathbf{a}$ but of unit length. Such a vector is called a unit vector and will be denoted by adding the suffix 1 to the symbol representing the vector. Thus $\mathbf{a}_1$ is a vector having the same direction as $\mathbf{a}$, but of unit length.

The length of a vector is termed its magnitude, size, or its absolute value. Sometimes, also, the term tensor is used. The magnitude of a vector $\mathbf{a}$ will be written a , using the same letter as that which denotes the vector but printed in *italic* type. It will be sometimes convenient also to denote the magnitude of $\mathbf{a}$ by adding the subscript 0 to $\mathbf{a}$ thus:

$$\mathbf{a}_0 = a.$$

The vector $\mathbf{a}$ then may be considered as one, a times as long as $\mathbf{a}_1$ and hence we may write:

$$\mathbf{a} = a \mathbf{a}_1 \text{ or } = \mathbf{a}_0 \mathbf{a}_1. \quad (1)$$

Any vector then may be represented by the product of its unit vector into its magnitude as in (1).

The expression $m \mathbf{a}$ denotes a vector m times as long as $\mathbf{a}$, having the same direction but m times its magnitude. The multiplier -1 from what has been said about negative vectors, reverses a vector.

Parallel vectors whatever their magnitude are said to be *collinear*.

Reciprocal Vector. The vector parallel to $\mathbf{a}$ but whose length is the reciprocal of the length of $\mathbf{a}$ is said to be the reciprocal of $\mathbf{a}$.

So that if

$$\begin{aligned} \mathbf{a} &= a \mathbf{a}_1 \\ \frac{1}{\mathbf{a}} &= \mathbf{a}^{-1} = \frac{\mathbf{a}_1}{a}. \end{aligned} \quad (2)$$

consideration of Fig. 3 will show :

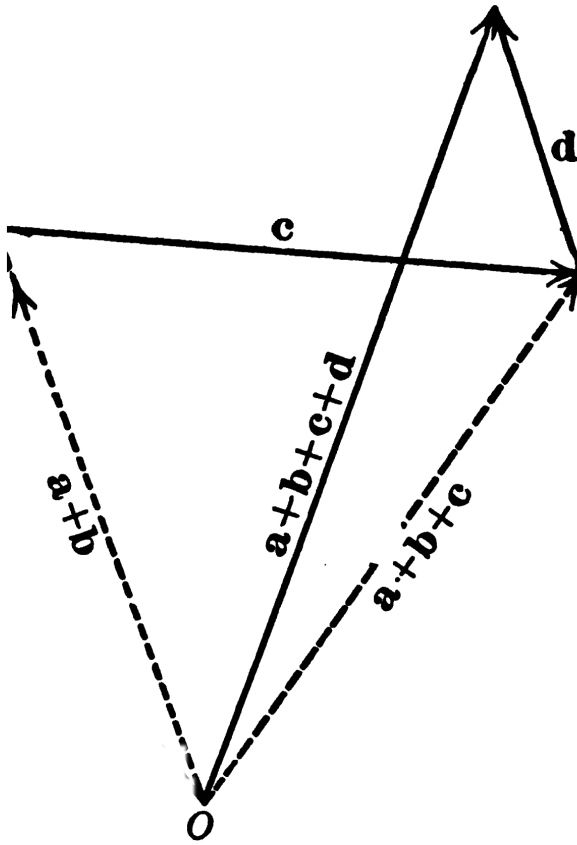


FIG. 3.

the origin of the first one to the terminus of the last one is the vector sum of them all.

A consideration of Fig. 3 will also show that the order in which they are taken is immaterial. The same construction then is used to find the sum of any number of vectors as is used in finding the resultant of the forces which would be represented by these vectors. Hence the importance of vectors in *mechanics*.

To subtract two vectors, add to the first the second one reversed. The extension of these rules to both positive and negative is obvious.

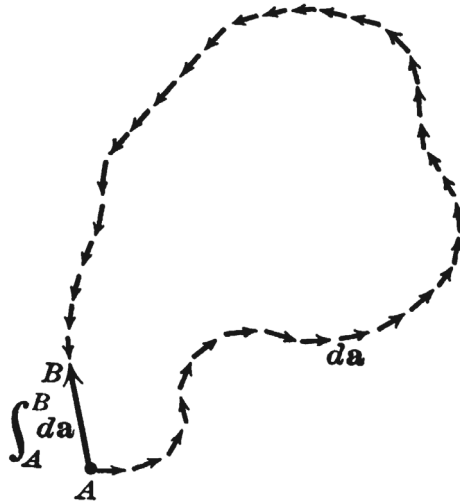


FIG. 4.

Vector Sum as an Integration. Any curve may be considered to be built up of an infinite number of infinitely short vectors, their directions being at every point along the tangent to the curve.

The sum of such a series of vectors differs in no way from the sum of a finite number of finite vectors. If da represents any one of these small vectors, then by adding them all the resultant AB is obtained. The operation of

adding this infinite number of infinitesimal vectors may be represented by an integration sign thus:

$$\vec{AB} = \int_A^B d\mathbf{a}. \quad (3)$$

If the curve is a closed one, whether a plane curve or not, then A and B coincide and $\vec{AB} = 0$ or $\int d\mathbf{a}$ around a closed path is zero.

Scalar and Vector Fields and their Addition.

5. Point-Function. Definition of Lamé. If for every position of a point in a region of space a quantity has one or more definite values assigned to it, it is said to be a function of the point, or more concisely, a point-function. We may have both scalar and vector point-functions.

As an **Example of a Scalar Point-Function**, consider the potential at any point due to any distribution of matter M_1 and let its value be V_1 . Now consider the potential at the same point P due to any other distribution of matter M_2 and let its value be V_2 . Then the potential at P due to both masses together is simply $V_1 + V_2$. This value is found by adding together the two scalars V_1 and V_2 .*

* Perhaps the following example of scalar field will be clearer to some minds. Consider a point P and let it be illuminated by a source of light M_1 . Evidently every point in the vicinity of the source is illuminated to a greater or lesser extent according to its distance from the source. The illumination or intensity of light at all points of the space considered may be represented by a scalar point-function. Let now another source of light M_2 be brought into the space under consideration. This source produces a certain intensity of illumination at every point of the space, of course including the point P . The total amount of illumination now received at the point P is the scalar sum of the amounts it receives from each individual source. This is true of every other point in the field. So that in general in order to find the illumination at any point due to separate sources, one simply adds the

Practical Definition of Continuity of a Scalar Point Function. If, as we go from any point in space to any near adjacent point, the magnitude of the scalar point-function

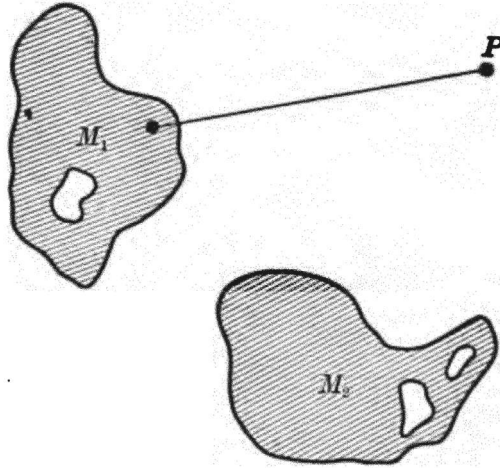


FIG. 5.

suffers no abrupt change, the function is said to be continuous.

As an **Example of a Vector Point-Function** consider the force of attraction at any point P due to the attraction of the mass M_1 . This force is evidently a vector quantity, as it has a definite magnitude and a definite direction, so that its representation requires the use of a vector at P ; let F_1 be this vector. Similarly let F_2 be the vector representing the force at P due to the matter M_2 . Then the force at P due to the combined action of M_1 and M_2 is the vector sum of F_1 and F_2 and must be obtained by the laws of vector addition; *i.e.*, the parallelogram law. If we go from the point P to another point Q in space the magnitudes and directions of the separate intensities at the point due to the separate sources respectively. This constitutes an addition of scalar fields. The fields are here *scalar* fields because we are considering only the *amounts* of the illumination received at any point.